Operations with functions

If f, g are two functions and c is a fixed real number, then

$$(f+g)(x) = f(x) + g(x)$$

$$(f-g)(x) = f(x) - g(x)$$

$$(c \cdot f)(x) = c \cdot f(x)$$

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$ (assume $g(x) \neq 0$ for any x in domain of $g, \frac{f}{g}$)

COMPOSITE FUNCTIONS

Create a new function by applying two functions back to back

__Ex__

$$\begin{tikzcd} x \arrow[r, bend left, "g"] \arrow[rr, bend right, "h"'] & x^2 - x + 2 \arrow[r, bend left, "f"] & (x^2 - x + 2)^2 \end{tikzcd}$$

$g(x) = x^2 - x + 2$ $\quad$ $f(x) = x^2$

$h(x) = f(g(x))$

$h(x) = (f \circ g)(x)$

"f composed with g of x"

Def^n: Suppose f and g are (real-valued) functions.

The composition of f with g, denoted $f \circ g$ is the function

$$(f \circ g)(x) = f(g(x))$$

which is defined for all values of x in the domain of g for which $g(x)$ is in the domain of f.

Note: order of composition matters! In general $f \circ g \neq g \circ f$

In the example above, $(f \circ g)(x) = (x^2 - x + 2)^2$

$$(g \circ f)(x) = (x^2)^2 - (x^2) + 2$$
$$= x^4 - x^2 + 2$$

Example: If $f(x) = x+1$ and $g(x) = \frac{1}{x-2}$, then

$$* \quad (f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x-2}\right) = \underbrace{\left(\frac{1}{x-2}\right) + 1}_{\text{applied the assignment rule for } f} = \frac{x-1}{x-2}$$

applied assignment rule for g

Domain of g is $\{x \in \mathbb{R} \mid x \neq 2\}$ (or $\mathbb{R} \setminus \{2\}$)

$(f \circ g)(x)$ is well defined for all $\mathbb{R} \setminus \{2\}$

$$* \quad (g \circ f)(x) = g(f(x)) \underset{\substack{\text{rule for } f \\ \text{(formula for } f)}}{\xrightarrow{\text{apply assignment}}} = g(x+1) \underset{\substack{\text{rule for } g \\ \text{(formula for } g)}}{\xrightarrow{\text{apply assignment}}} = \frac{1}{(x+1)-2} = \frac{1}{x-1}$$

$(g \circ f)(x)$ is well defined for all $\mathbb{R} \setminus \{1\}$

(or all values of x such that $f(x) \neq 2$)

Even and Odd functions

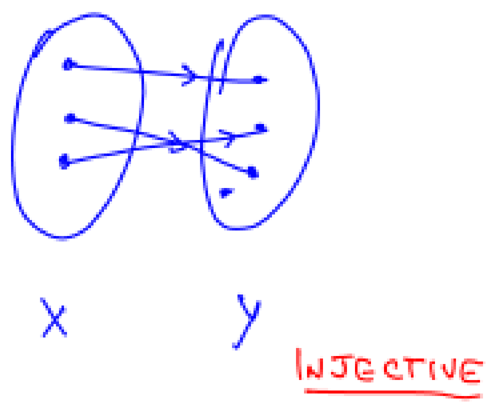
A function f is called <u>even</u> if for any x in the domain of f, $-x$ also belongs to the domain of f, and $\underline{f(-x) = f(x)}$.

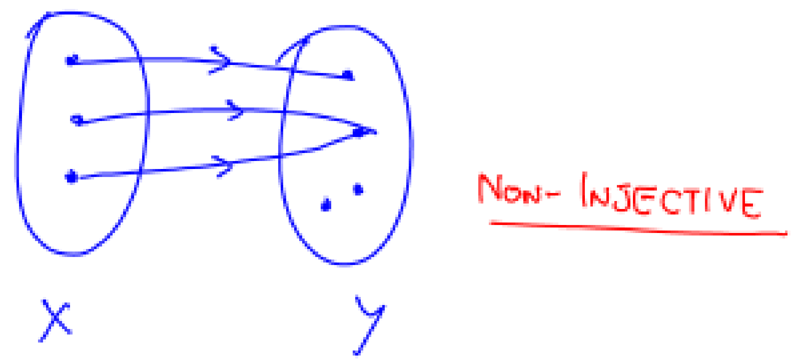
A function f is called <u>odd</u> if for any x in the domain — " —

— " — and $\underline{f(-x) = -f(x)}$.

Injective or One-to-One function

A function $f : X \to Y$ is called injective or one-to one if, for all $x_1 \in X$, $x_2 \in X$ with $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$.





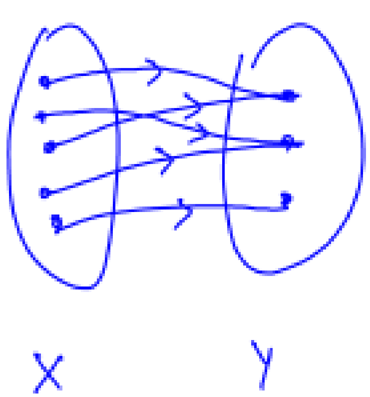
Ex:

$f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 5x - 2$ is injective

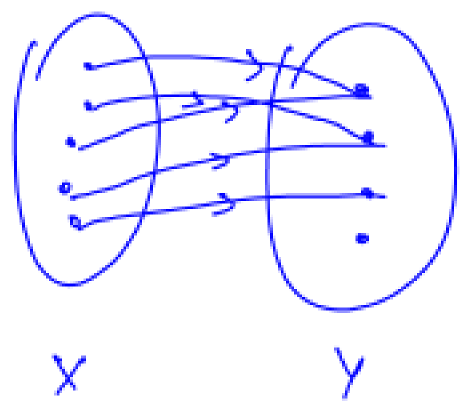
$f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ is not injective Why? $f(1) = f(-1) = 1$.

Surjective or Onto functions

A function $f: X \to Y$ is called surjective or onto if for all $y \in Y$ there exists $x \in X$ such that $f(x) = y$.



SURJECTIVE



NON-SURJECTIVE

Example: $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3$ is surjective

$f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ is not surjective

Why? there is no $x \in \mathbb{R}$ such that $f(x) = -1$.

$f: \mathbb{R} \to A$ with $f(x) = x^2$ is surjective when $A = \{x \in \mathbb{R} \mid x \geq 0\}$

Inverse of a function

A function $f: A \to B$ is said to be invertible provided there exists a function $g: B \to A$ satisfying

(i) $g(f(x)) = x$ for all x in A.

(ii) $f(g(y)) = y$ for all y in B.

g is called the inverse of f and is denoted by f^{-1}.