LECTURE 04

01/19/2017

RECAP:

- SEQUENCES
  - types of sequences (arithmetic, geometric)
  - direct / recursive formulas
- CONVERGENT SEQUENCES
  - definition of convergent sequence
  - divergent sequences
  - bounded / unbounded sequences
- PROPERTIES OF CONVERGENT SEQUENCES
  - sandwich theorem
  - limit arithmetic

Note:

- convergent sequence $\Rightarrow$ bounded sequence, but not all bounded sequences are convergent
- to show that a sequence is bounded, you don't need to show that it is convergent. (You should be able to use some algebra to find an M that satisfies the definition) Ex: $x_n = \frac{n}{n+3}$; we have $\frac{n}{n+3} \leq \frac{n}{n} = 1$

choice of M

Note:

- sequences are particularly useful in computing approximations to (real) irrational numbers

  (recall the week 1 exercise used to compute square root approximations)

- can think of real numbers (in particular, irrational numbers such as π, $\sqrt{3}$) as limits of sequences of rational numbers.

- for a real number x, we can select two sequences $\{y_n\}_n$ and $\{z_n\}_n$ with

  - y_n, z_n are rational number for all n
  - y_n is non decreasing $(y_1 \leq y_2 \leq y_3 \leq \cdots)$
    z_n is non increasing $(z_1 \geq z_2 \geq z_3 \geq \cdots)$
  - $y_n \leq x \leq z_n$ for all n
  - $\lim_{n \to \infty} y_n = \lim_{n \to \infty} z_n = x$.

(see week 2 exercise on approximating area under curves)

Infinite Series

Def^n: An <u>infinite series</u> is an expression of the form

$$x_1 + x_2 + x_3 + \cdots,$$

where the numbers $x_1, x_2, x_3, \ldots$ are the <u>terms</u> of the series. The sequence $\{s_n\}_n$ of <u>partial sums</u> is

$$s_1 = x_1$$

$$s_2 = x_1 + x_2$$

$$s_3 = x_1 + x_2 + x_3$$

$$\vdots$$

$$s_n = x_1 + x_2 + x_3 + \cdots + x_n = \sum_{i=1}^{n} x_i \quad \text{for each } n \geq 1.$$

If the sequence $\{s_n\}_n$ converges to a real number L (i.e., $L = \lim_{n\to\infty} s_n$), then the above series is <u>convergent</u>, and L is called the <u>sum</u> of the series.

We write

$$x_1 + x_2 + x_3 + \cdots = \sum_{i=1}^{\infty} x_i = L.$$

Otherwise, we say that the series is <u>divergent</u>.

<u>Example:</u> (Recall arithmetic sequences)

Consider the infinite series $a_1 + a_2 + a_3 + \ldots\ldots$ where $a_n = \underbrace{a_1}_{\text{first term (given)}} + (n-1)\underbrace{d}_{\text{common difference}}$

Consider partial sums of this series

$$S_n = a_1 + a_2 + \ldots\ldots + a_n$$

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots\ldots + (a_1 + (n-1)d) \quad \text{— ①}$$

we may also write this as

$$S_n = (a_n - (n-1)d) + (a_n - (n-2)d) + \ldots\ldots + (a_n - d) + a_n \quad \text{— ②}$$

note:

$$a_n - (n-1)d = (a_1 + \cancel{(n-1)}d) - \cancel{(n-1)}d = a_1$$

adding ① and ②,

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + \ldots\ldots + (a_1 + a_n) + (a_1 + a_n)$$

$$= n(a_1 + a_n)$$

therefore, $\boxed{S_n = \frac{n}{2}(a_1 + a_n)}$

Substituting for a_n, we get

$$\boxed{S_n = na_1 + \frac{n(n-1)}{2}d}$$

Geometric Series

Consider an infinite series whose terms form a geometric sequence with common ratio r. Such a series is called a geometric series of ratio r.

If a is the first term of the series, then the series may be written as

$$a + ar + ar^2 + \dots\dots$$

Geometric Series convergence

Theorem: For any nonzero real number a, the geometric series

$$a + ar + ar^2 + \dots\dots$$

converges to the sum $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.

Note: partial sum of a geometric series

we have $S_n = \sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + \dots\dots + ar^{n-1}$

multiplying both sides by $(1-r)$

$$(1-r)S_n = (1-r)(a + ar + ar^2 + \dots\dots + ar^{n-1})$$

$$- (a + \cancel{ar} + \cancel{ar^2} + \dots\dots + \cancel{ar^{n-1}}) - (\cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \dots\dots + ar^n)$$

$$= a - ar^n$$

or, $\boxed{S_n = \dfrac{a(1-r^n)}{1-r}}$ (when $r \neq 1$)

note: when $r=1$, $S_n = a + a + \dots + a = na$

Sum of a geometric series

$$L = \sum_{k=1}^{\infty} ar^{k-1} = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \lim_{n\to\infty} \frac{r^n}{1-r}$$

If $|r| < 1$, $r^n \to 0$ as $n \to \infty$.

Therefore $L = \dfrac{a}{1-r} - 0 = \dfrac{a}{1-r}$.

EXAMPLES

① Decide whether the geometric series converges or diverges. Justify your answer. If the series converges, compute its sum

(a) $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \cdots\cdots$

Here, we have a geometric series with first term $a = \frac{1}{3}$ and common ratio $r = -\frac{1}{3}$. Since $|r| = \left|-\frac{1}{3}\right| = \frac{1}{3} < 1$, the convergence theorem for geometric series guarantees convergence to the sum

$$L = \frac{a}{1-r} = \frac{1/3}{1-(-1/3)} = \frac{1/3}{4/3} = 1/4.$$

(b) $1 + \frac{\pi}{3} + \frac{\pi^2}{9} + \frac{\pi^3}{27} + \frac{\pi^4}{81} + \cdots\cdots$

For this geometric series, $a = 1$ and common ratio $r = \pi/3$.

Since $|r| = \pi/3 > 1$, this series diverges.