LECTURE 03 | 01/17/2017

INCREASING / DECREASING MONOTONIC SEQUENCES

A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be <u>increasing</u> if

$$x_n < x_{n+1} \quad \text{for all } n \geq 1$$

if $\{x_n\}_{n\in\mathbb{N}}$ satisfies $x_n \leq x_{n+1}$ for all $n \geq 1$, it is called <u>non-decreasing</u>

<u>Example</u>: $\{2n+5\}_{n\in\mathbb{N}}$

A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be <u>decreasing</u> if $x_n > x_{n+1}$ for all $n \geq 1$.

If $\{x_n\}_{n\in\mathbb{N}}$ satisfies $x_n \geq x_{n+1}$ for all $n \geq 1$, then it is called <u>non-increasing</u>.

<u>Example</u>: $\{1/3^n\}_{n\in\mathbb{N}}$

A sequence is called <u>monotonic</u> if it is non-decreasing or non-increasing

PROPERTIES OF SEQUENCES

The following properties are useful in studying the convergence of sequences.

(P1) The sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to L if and only if the sequence $\{x_n\}_{n\geq K}$ converges to L, where K is a positive integer.

(i.e., convergence is not affected by changing/omitting a finite number of terms in the sequence)

(P2) A convergent sequence has a unique limit

(P3) If a sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent, then it is <u>bounded</u>

Def^n: A sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded if there exists a positive real number M such that $|x_n| \leq M$ for all $n\in\mathbb{N}$

(P4) A non-decreasing (non-increasing) sequence is convergent if and only if it is bounded.

(P5) Here $\{x_n\}_n$, $\{y_n\}_n$ are two convergent sequences and λ is a real number.

(i) the sequence $\{x_n + y_n\}_n$ is convergent with

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

(ii) the sequence $\{x_n - y_n\}_n$ is convergent with

$$\lim_{n\to\infty} (x_n - y_n) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n.$$

(iii) the sequence $\{c \cdot y_n\}_n$ is convergent with

$$\lim_{n\to\infty} (c \cdot y_n) = c \cdot \lim_{n\to\infty} y_n.$$

(iv) the sequence $\{x_n \cdot y_n\}_n$ is convergent with

$$\lim_{n\to\infty} (x_n \cdot y_n) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n$$

(v) If $y_n \neq 0$ for all n and $\lim\limits_{n\to\infty} y_n \neq 0$, then the sequence $\left\{\frac{x_n}{y_n}\right\}_n$ is convergent with

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}.$$

(P6) (Sandwich Theorem) Suppose the sequences $\{x_n\}_n$, $\{y_n\}_n$ and $\{z_n\}_n$ are such that

(i) $x_n \leq y_n \leq z_n$ for all n, and

(ii) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = L$ for some $L \in \mathbb{R}$,

then the sequence $\{y_n\}_n$ is convergent and $\lim\limits_{n\to\infty} y_n = L$.

(P7) Let $\{x_n\}_n$ and $\{y_n\}_n$ be two sequences such that $x_n \leq y_n$ for all n. Then,

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

Examples:

Decide if the sequence is convergent or divergent. If convergent, determine its limit

> Note: can use this without justification
>
> $$\lim_{n\to\infty} \frac{1}{n} = 0.$$

(a) $x_n = \dfrac{1}{n^3}$

Applying the sandwich theorem, $0 \leq x_n \leq \dfrac{1}{n}$ for all $n \geq 1$.

Since $\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} \frac{1}{n} = 0$, we have $\lim_{n\to\infty} x_n = 0$.

(b) $y_n = \dfrac{3n+1}{7n-4}$

Factoring and simplifying, $\displaystyle\lim_{n\to\infty} \frac{3n+1}{7n-4} = \lim_{n\to\infty} \frac{(3+1/n)}{(7-4/n)}$ (factoring out n)

property P5 (i)

property P5 (v) +

property P5 (ii)

$$= \frac{\lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{1}{n}}{\lim_{n\to\infty} 7 - \lim_{n\to\infty} \frac{4}{n}} = \frac{3}{7}$$