Fast Robust Phase Retrieval from Local Correlation Measurements

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The Phase Retrieval Problem

$$\text{find}^1 \quad \mathbf{x} \in \mathbb{C}^d \quad \text{given} \quad |M\mathbf{x}|^2 + \mathbf{n} = \mathbf{b} \in \mathbb{R}^D,$$

where

- $\mathbf{b} \in \mathbb{R}^D$ denotes the phaseless (or magnitude-only) measurements,
- $M \in \mathbb{C}^{D \times d}$ is a measurement matrix associated with these measurements, and
- $\mathbf{n} \in \mathbb{R}^D$ is measurement noise.

Let $\mathcal{A} : \mathbb{R}^D \to \mathbb{C}^d$ denote the recovery method. The phase retrieval problem involves designing measurement matrix and recovery method pairs, $(M, \mathcal{A})$.

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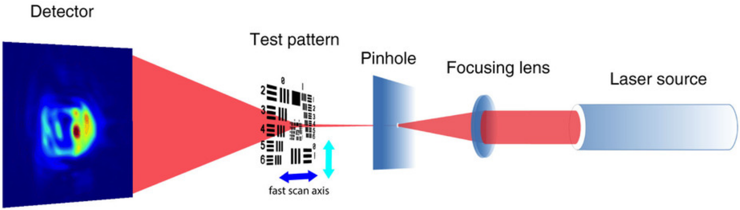
[1] (upto a global phase offset)

Motivating Applications



From Huang, Xiaojing, et al. "Fly-scan ptychography." *Scientific Reports* 5 (2015).

The Phase Retrieval problem arises in many molecular imaging modalities, including

- X-ray crystallography
- Ptychography

Other applications can be found in optics, astronomy and speech processing.

Existing Computational Approaches

- Alternating projection methods
  [Fienup, 1978], [Fannjiang and Liao, 2012], ...

- Methods based on semidefinite programming
  PhaseLift [Candes et al., 2012], PhaseCut [Waldspurger et al., 2012], ...

- Others
  - Frame-theoretic, graph based algorithms [Alexeev et al., 2014]
  - (Stochastic) gradient descent [Candes et al., 2014]

... and variants for sparse and/or structured signal models.

Existing Computational Approaches

- Alternating projection methods – No recovery guarantees
  [Fienup, 1978], [Fannjiang and Liao, 2012], . . .
- Methods based on semidefinite programming – Expensive
  PhaseLift [Candes et al., 2012], PhaseCut [Waldspurger et al., 2012], . . .
- Others
  - Frame-theoretic, graph based algorithms [Alexeev et al., 2014]
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Most methods require random measurement constructions.

Today. . .

- We discuss a recently introduced essentially linear-time phase retrieval algorithm based on (deterministic[2]) local correlation measurement constructions.

- We provide rigorous theoretical recovery guarantees and present numerical results showing the accuracy, efficiency and robustness of the method.

[2] for a large class of real-world signals

Outline

Key Components

1. **Local Measurements**: Each measurement provides information about some *local* region of $\mathbf{x}$.

2. **Local Lifting**: Use compactly supported masks and correlation measurements to obtain phase difference estimates.

$$|\mathtt{Corr}(\mathbf{m}_i, \mathbf{x})|^2 \xrightarrow[\text{linear system}]{\text{solve}} \{x_j x_k^*\}_{|j-k \mod d < \delta|}$$

   - $\mathbf{m}_i$ is a mask or window function with δ non-zero entries.
   - $x_j x_k^*$ provides (scaled) phase difference between x_j and x_k.

3. **Angular Synchronization**: Use the phase differences to obtain the phases of the unknown signal.

$$\{x_j x_k^*\}_{|j-k \mod d < \delta|} \xrightarrow[\text{synchronization}]{\text{angular}} \{x_j\}_{j=1}^{d}$$

Constraint on $\mathbf{x}$: We require $\mathbf{x}$ to be "flat".
(At most δ consecutive entries in $\mathbf{x}$ with magnitude $< \frac{\|\mathbf{x}\|_2}{2\sqrt{d}}$)

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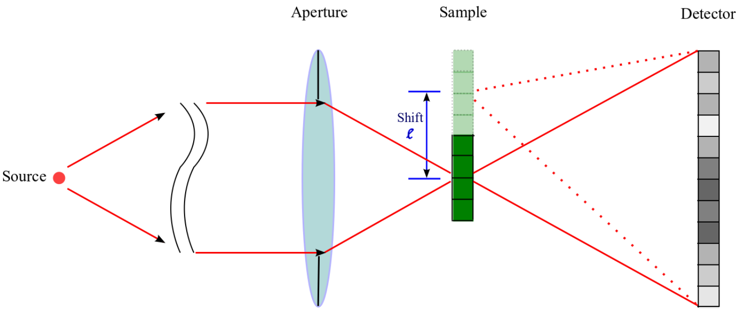
Measurement Constructions



Adapted from Huang et al. "Fly-scan ptychography." *Scientific Reports* 5 (2015).

- We consider measurements motivated by **Ptychographic** molecular imaging.
- Measurements are **local**; the full reconstruction is obtained by imaging *shifts* of the specimen.

Model Problem

Recover an unknown vector $\mathbf{x} \in \mathbb{C}^4$ from noiseless measurements

$$\mathbf{y} = |M\mathbf{x}|^2,$$

where $\mathbf{y} \in \mathbb{R}^{12}$ and $M \in \mathbb{C}^{12\times 4}$ has the following structure:

$$M = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}, \quad M_i = \begin{bmatrix} (\mathbf{m}_i)_1 & (\mathbf{m}_i)_2 & 0 & 0 \\ 0 & (\mathbf{m}_i)_1 & (\mathbf{m}_i)_2 & 0 \\ 0 & 0 & (\mathbf{m}_i)_1 & (\mathbf{m}_i)_2 \\ (\mathbf{m}_i)_2 & 0 & 0 & (\mathbf{m}_i)_1 \end{bmatrix}.$$

Here, $\mathbf{m}_{\{1,2,3\}} \in \mathbb{C}^4$ are masks with *local support* (with $\delta = 2$ non-zero entries).

Local Correlation Measurements

These correspond to *local correlation measurements*

$$(\mathbf{y}_\ell)_i = \left| \sum_{k=1}^{\delta=2} (\mathbf{m}_\ell)_k \cdot x_{i+k-1} \right|^2, \qquad (\ell, i) \in \{1,2,3\} \times \{1,2,3,4\}$$

$$= \sum_{j,k=1}^{\delta} (\mathbf{m}_\ell)_j \, (\mathbf{m}_\ell)_k^* \, x_{i+j-1} \, x_{i+k-1}^* := \sum_{j,k=1}^{\delta} (\mathbf{m}_\ell)_{j,k} \, x_{i+j-1} \, x_{i+k-1}^*.$$

This is a linear system for the phase differences $\{x_j x_k^*\}$!

Note: the masks $\mathbf{m}_{\{1,2,3\}}$ (which are related to the aperture transmission function of the imaging system) are known - either by design or through calibration.

Solving for Phase Differences

Writing out the correlation sum, we obtain the linear system

$$M'\,\mathbf{z} = \widetilde{\mathbf{b}},$$

where

$$\mathbf{z} = \left[|x_1|^2 \;\; x_1x_2^* \;\; x_2x_1^* \;\; |x_2|^2 \;\; x_2x_3^* \;\; x_3x_2^* \;\; |x_3|^2 \;\; x_3x_4^* \;\; x_4x_3^* \;\; |x_4|^2 \;\; x_4x_1^* \;\; x_1x_4^*\right]^T,$$

$$\widetilde{\mathbf{b}} = \left[(y_1)_1 \;\; (y_2)_1 \;\; (y_3)_1 \;\; (y_1)_2 \;\; (y_2)_2 \;\; (y_3)_2 \;\; (y_1)_3 \;\; (y_2)_3 \;\; (y_3)_3 \;\; (y_1)_4 \;\; (y_2)_4 \;\; (y_3)_4\right]^T,$$

$$M' = \begin{bmatrix}
(\mathbf{m}_1)_{1,1} & (\mathbf{m}_1)_{1,2} & (\mathbf{m}_1)_{2,1} & (\mathbf{m}_1)_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(\mathbf{m}_2)_{1,1} & (\mathbf{m}_2)_{1,2} & (\mathbf{m}_2)_{2,1} & (\mathbf{m}_2)_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(\mathbf{m}_3)_{1,1} & (\mathbf{m}_3)_{1,2} & (\mathbf{m}_3)_{2,1} & (\mathbf{m}_3)_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (\mathbf{m}_1)_{1,1} & (\mathbf{m}_1)_{1,2} & (\mathbf{m}_1)_{2,1} & (\mathbf{m}_1)_{2,2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (\mathbf{m}_2)_{1,1} & (\mathbf{m}_2)_{1,2} & (\mathbf{m}_2)_{2,1} & (\mathbf{m}_2)_{2,2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (\mathbf{m}_3)_{1,1} & (\mathbf{m}_3)_{1,2} & (\mathbf{m}_3)_{2,1} & (\mathbf{m}_3)_{2,2} & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & (\mathbf{m}_3)_{1,1} & (\mathbf{m}_3)_{1,2} & (\mathbf{m}_3)_{2,1} & (\mathbf{m}_3)_{2,2} & 0 & 0 \\
(\mathbf{m}_1)_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\mathbf{m}_1)_{1,1} & (\mathbf{m}_1)_{1,2} & (\mathbf{m}_1)_{2,1} \\
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\end{bmatrix}$$

Back to our Example . . .

$$\left[|x_1|^2 \; x_1x_2^* \; x_2x_1^* \; |x_2|^2 \; x_2x_3^* \; x_3x_2^* \; |x_3|^2 \; x_3x_4^* \; x_4x_3^* \; |x_4|^2 \; x_4x_1^* \; x_1x_4^*\right]^T$$

$\downarrow$ (re-arrange)

$$\begin{bmatrix} |x_1|^2 & x_1x_2^* & 0 & x_1x_4^* \\ x_2x_1^* & |x_2|^2 & x_2x_3^* & 0 \\ 0 & x_3x_2^* & |x_3|^2 & x_3x_4^* \\ x_4x_1^* & 0 & x_4x_3^* & |x_4|^2 \end{bmatrix}$$ ($2\delta - 1$ entries in band)

$\downarrow$ (normalize)

$$\begin{bmatrix} 1 & e^{\mathbb{i}(\phi_1-\phi_2)} & 0 & e^{\mathbb{i}(\phi_1-\phi_4)} \\ e^{\mathbb{i}(\phi_2-\phi_1)} & 1 & e^{\mathbb{i}(\phi_2-\phi_3)} & 0 \\ 0 & e^{\mathbb{i}(\phi_3-\phi_2)} & 1 & e^{\mathbb{i}(\phi_3-\phi_4)} \\ e^{\mathbb{i}(\phi_4-\phi_1)} & 0 & e^{\mathbb{i}(\phi_4-\phi_3)} & 1 \end{bmatrix}$$

$\downarrow$ (angular synchronization)

$$\phi_1, \phi_2, \phi_3, \phi_4$$

(Signal Reconstruction) $\left[|x_1|e^{\mathbb{i}\,\phi_1} \quad |x_2|e^{\mathbb{i}\,\phi_2} \quad |x_3|e^{\mathbb{i}\,\phi_3} \quad |x_4|e^{\mathbb{i}\,\phi_4}\right]^T$

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$$\left[|x_1|^2 \; x_1x_2^* \; x_2x_1^* \; |x_2|^2 \; x_2x_3^* \; x_3x_2^* \; |x_3|^2 \; x_3x_4^* \; x_4x_3^* \; |x_4|^2 \; x_4x_1^* \; x_1x_4^*\right]^T$$

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The Angular Synchronization Problem

The Angular Synchronization Problem

Estimate d unknown angles $\phi_1, \phi_2, \ldots, \phi_d \in [0, 2\pi)$ from noisy and possibly incomplete measurements of their differences,

$$\phi_{i,j} := \phi_i - \phi_j \mod 2\pi.$$

- Several possible approaches: eigenvector methods, semidefinite programming ...
- Today: Greedy angular synchronization

Greedy Angular Synchronization

1. Set the largest magnitude component to have zero phase angle; i.e.,

$$\arg(x_j) = 0, \qquad j = \underset{i}{\operatorname{argmax}} \, |x_i|^2.$$

2. Use this entry to set the phase angles of its δ neighboring entries; i.e.,

$$\arg(x_k) = \arg(x_j) - \phi_{j,k}, \qquad |j - k \bmod d| < \delta.$$

3. Use the next largest magnitude component from these δ entries and repeat the process.

Greedy Angular Synchronization

$$\left[|x_1|^2 \; x_1x_2^* \; x_2x_1^* \; |x_2|^2 \; x_2x_3^* \; x_3x_2^* \; |x_3|^2 \; x_3x_4^* \; x_4x_3^* \; |x_4|^2 \; x_4x_1^* \; x_1x_4^*\right]^T$$

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Greedy Angular Synchronization

Applying this to our example problem...

- Assume, without loss of generality, that $|x_1| \geq |x_i|$, $i \in \{2, 3, 4\}$.

1. We start by setting[3] $\arg(x_1) = 0$.
2. We may now set the phase of x_2 and x_4 using the estimated phase differences $\phi_{1,2}$ and $\phi_{1,4}$ respectively; i.e.,

$$\arg(x_2) = \arg(x_1) - \phi_{1,2}, \quad \arg(x_4) = \arg(x_1) - \phi_{1,4}.$$

3. Similarly, we next set $\arg(x_3) = \arg(x_2) - \phi_{2,3}$, thereby recovering all of the entries' unknown phases.

[3] Recall that we can only recover $\mathbf{x}$ up to an unknown global phase factor which, in this case, will be the true phase of x_1.

Outline

Block-Circulant Matrix: Condition Number Bounds

Theorem (Iwen, V., Wang 2015)

Choose entries of the measurement mask $(\mathbf{m}_i)$ *as follows:*

$$(\mathbf{m}_i)_\ell = \begin{cases} \frac{\mathbb{e}^{-\ell/a}}{\sqrt[4]{2\delta-1}} \cdot \mathbb{e}^{\frac{2\pi \mathbb{i} \cdot i \cdot \ell}{2\delta-1}}, & \ell \le \delta \\ 0, & \ell > \delta \end{cases}, \qquad \begin{array}{l} a := \max\left\{4, \frac{\delta-1}{2}\right\}, \\ i = 1, 2, \ldots, N. \end{array}$$

Then, the resulting system matrix for the phase differences, M', *has condition number*

$$\kappa(M') < \max\left\{144\mathbb{e}^2, \frac{9\mathbb{e}^2}{4} \cdot (\delta - 1)^2\right\}.$$

- Deterministic (windowed DFT-type) measurement masks!
- δ is typically chosen to be $c \log_2 d$ with c small $(2\text{–}3)$.
- Extensions: oversampling, random masks ....

Recovery Guarantee – Non-Sparse ("Flat") Signals

Theorem (Iwen, V., Wang 2015)

There exist fixed universal constants $C, C' \in \mathbb{R}^+$ *such that following holds: Let* $M \in \mathbb{C}^{D\times d}$ *be defined as in the previous slide, and suppose that* $\mathbf{x} \in \mathbb{C}^d$ *is non-sparse*[a] *with* $d > 2$ *and* $\|\mathbf{x}\|_2^2 \geq C\ (\delta - 1)d^2\ \|\mathbf{n}\|_2$*. Then, the proposed algorithm is guaranteed to recover an* $\tilde{\mathbf{x}} \in \mathbb{C}^d$ *with*

$$\min_{\theta\in[0,2\pi)} \left\|\mathbf{x} - \mathbb{e}^{\mathbb{i}\theta}\tilde{\mathbf{x}}\right\|_2^2 \ \leq\ C' d^2(\delta - 1)\|\mathbf{n}\|_2$$

when given arbitrarily noisy input measurements $\mathbf{b} = |M\mathbf{x}|^2 + \mathbf{n} \in \mathbb{R}^D$*. Furthermore, the algorithm requires just* $\mathcal{O}(\delta \cdot d \log d)$ *operations for this choice of* $M \in \mathbb{C}^{D\times d}$*.*

[a] does not have more than $\lfloor(\delta - 3)/2\rfloor$ consecutive zeros or small entries; see preprint for details.

Recovery Guarantee - Arbitrary Signals

Theorem (Iwen, V., Wang 2015)

Let $\mathbf{x} \in \mathbb{C}^d$ *with* d *sufficiently large have* $\|\mathbf{x}\|_2^2 \geq C\ (d\ \ln d)^2 \ln^3(\ln d)\ \|\mathbf{n}\|_2$.[a] *Then, one can select a random measurement matrix* $\tilde{M} \in \mathbb{C}^{D \times d}$ *such that the following holds with probability at least* $1 - \frac{1}{C' \cdot \ln^2(d) \cdot \ln^3(\ln d)}$*: the proposed algorithm will recover an* $\tilde{\mathbf{x}} \in \mathbb{C}^d$ *with*

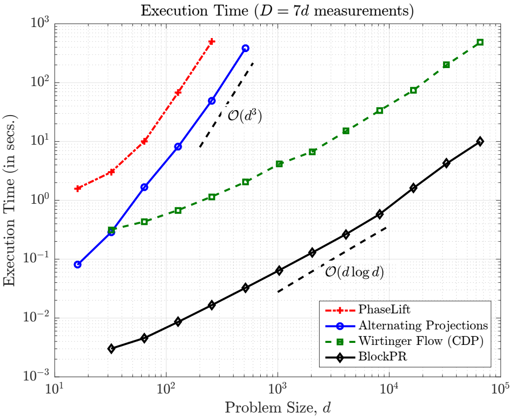
$$\min_{\theta \in [0, 2\pi)} \left\| \mathbf{x} - \mathrm{e}^{\mathrm{i}\theta} \tilde{\mathbf{x}} \right\|_2^2 \ \leq\ C''(d\ \ln d)^2 \ln^3(\ln d) \|\mathbf{n}\|_2$$

when given arbitrarily noisy input measurements $\mathbf{b} = |M\mathbf{x}|^2 + \mathbf{n} \in \mathbb{R}^D$*. Here* D *can be chosen to be* $\mathcal{O}(d \cdot \ln^2(d) \cdot \ln^3(\ln d))$*. Furthermore, the algorithm will run in* $\mathcal{O}(d \cdot \ln^3(d) \cdot \ln^3(\ln d))$*-time.*

[a] Herein $C, C', C'' \in \mathbb{R}^+$ are all fixed and absolute constants.

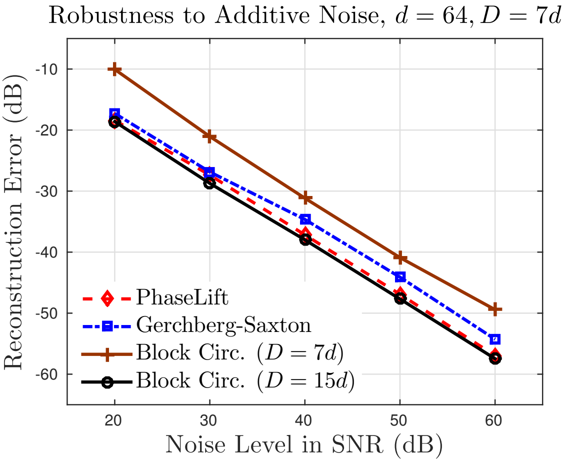
Outline

Efficiency



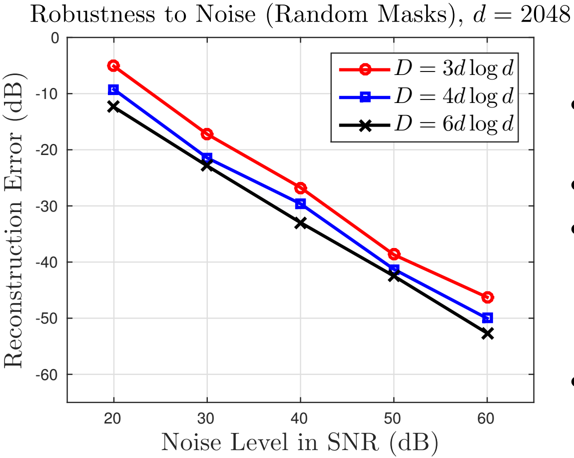
- iid Complex Gaussian test signal
- Averaged over 100 trials
- Simulations performed in Matlab on a laptop computer with 4GB RAM

Robustness



- iid complex Gaussian signal
- $d = 64$
- $7d$ measurements
- Deterministic (windowed Fourier-like) measurements

Robustness



- iid complex Gaussian signal
- $d = 2048$
- Not feasible with SDP-based methods such as PhaseLift on a laptop in Matlab
- Random measurements

In Summary...

- BlockPR allows for essentially linear-time robust phase retrieval from local correlation measurement constructions.
- Deterministic measurements for flat vectors.
- **First** known global *robust* recovery guarantee for phase retrieval from local correlation (ptychographic) measurements.

Current and Future Directions

- (Sublinear-time) compressive phase retrieval
- Improved angular synchronization frameworks
- Extensions to 2D and Ptychography

Publications/Preprints/Code

This Talk

Mark Iwen, A. Viswanathan and Yang Wang. "Fast Phase Retrieval from Local Correlation Measurements." arXiv:1501.02377, 2015.

Code: https://bitbucket.org/charms/blockpr

Related Work

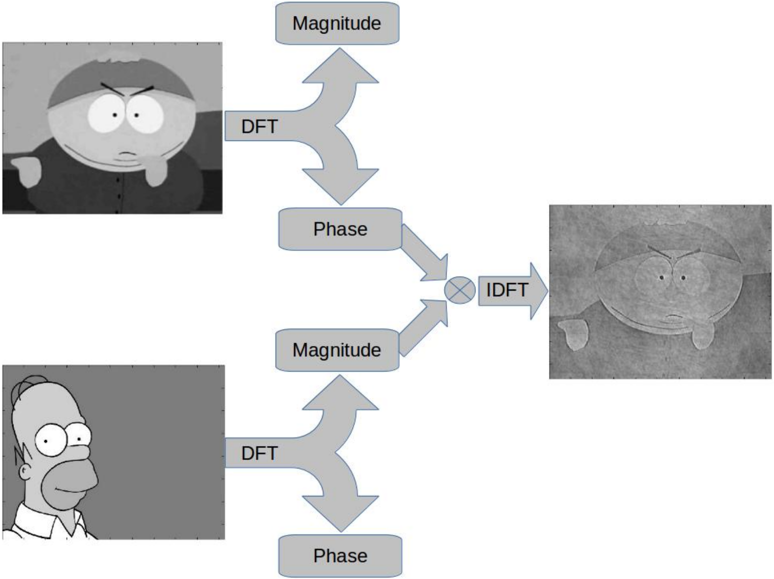
M. Iwen, A. Viswanathan, and Y. Wang. "Robust Sparse Phase Retrieval Made Easy." (in press) ACHA, 2015. arXiv:1410.5295

A. Viswanathan and Mark Iwen. "Fast Angular Synchronization for Phase Retrieval via Incomplete Information." Proc. SPIE 9597, Wavelets+Sparsity XVI, 2015.

A. Viswanathan and Mark Iwen. "Fast Compressive Phase Retrieval." Asilomar Conf. Signals, Systems Computers, 2015.

Code: https://bitbucket.org/charms/sparsepr

Questions?



Appendix: Condition Number Proof Sketch

(Step 1) M' is block-circulant and therefore admits a unitary decomposition

$$U_{2\delta-1}^{*} M' U_{2\delta-1} = J = \text{blockdiag}\,(J_1, J_2, \ldots, J_d),$$

where $J_1, \ldots, J_d \in \mathbb{C}^{(2\delta-1)\times(2\delta-1)}$ are defined as

$$J_k := \sum_{l=1}^{\delta} M_l' \cdot \mathbb{e}^{\frac{2\pi \mathbb{i}\cdot(k-1)\cdot(l-1)}{d}}$$

and $U_\alpha \in \mathbb{C}^{\alpha d \times \alpha d}$ are unitary block Fourier matrices defined by

$$U_\alpha := \frac{1}{\sqrt{d}} \begin{pmatrix} I_\alpha & I_\alpha & & \ldots & I_\alpha \\ I_\alpha & I_\alpha \mathbb{e}^{\frac{2\pi\mathbb{i}}{d}} & & \ldots & I_\alpha \mathbb{e}^{\frac{2\pi\mathbb{i}\cdot(d-1)}{d}} \\ & & & \ddots & \\ I_\alpha & I_\alpha \mathbb{e}^{\frac{2\pi\mathbb{i}\cdot(d-2)}{d}} & & \ldots & I_\alpha \mathbb{e}^{\frac{2\pi\mathbb{i}\cdot(d-2)\cdot(d-1)}{d}} \\ I_\alpha & I_\alpha \mathbb{e}^{\frac{2\pi\mathbb{i}\cdot(d-1)}{d}} & & \ldots & I_\alpha \mathbb{e}^{\frac{2\pi\mathbb{i}\cdot(d-1)\cdot(d-1)}{d}} \end{pmatrix}.$$

Appendix: Condition Number Proof Sketch

(Step 2) For the prescribed structured measurements, evaluating J_k yields

$$J_k = F_{2\delta-1} \begin{pmatrix} s_{k,1} & 0 & \dots & 0 \\ 0 & s_{k,2} & 0 & \dots \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & s_{k,2\delta-1} \end{pmatrix},$$

where $F_\alpha \in \mathbb{C}^{\alpha\times\alpha}$ is the unitary $\alpha \times \alpha$ discrete Fourier transform matrix, and $\{s_{k,j}\}_{k\in\{1,2,\dots,d\}}^{j\in\{1,2,\dots,2\delta-1\}}$ can be explicitly evaluated.

Since $F_{2\delta-1}$ is unitary,

$$\min_{j\in[2\delta-1]} |s_{k,j}| \le \sigma_{2\delta-1}(J_k) \le \sigma_1(J_k) \le \max_{j\in[2\delta-1]} |s_{k,j}|.$$

Appendix: Condition Number Proof Sketch

(Step 3) Bound the maximum and minimum values of $|s_{k,j}|$ from above and below, respectively, over all $k \in \{1, 2, \ldots, d\}$ and $j \in \{1, 2, \ldots, 2\delta - 1\}$. Minimize upper bound w.r.t. a parameter.

(Step 4) Final result obtained using

$$\kappa\left(M'\right) = \frac{\sigma_1\left(M'\right)}{\sigma_D\left(M'\right)} = \frac{\sigma_1\left(J\right)}{\sigma_D\left(J\right)} \leq \frac{\max_{k \in \{1,2,\ldots,d\}} \sigma_1\left(J_k\right)}{\min_{k \in \{1,2,\ldots,d\}} \sigma_{2\delta-1}\left(J_k\right)}.$$

Appendix: Flattening "Non-Sparse" Vectors

- Recall: Due to compact support of our masks, only $\left\lfloor \frac{\delta-3}{2} \right\rfloor$-flat vectors can be recovered
- Arbitrary vectors can be "flattened" by multiplication with a random unitary matrix such as $W = PFB$, where
  - $P \in \{0,1\}^{d \times d}$ is a permutation matrix selected uniformly at random from the set of all $d \times d$ permutation matrices
  - F is the unitary $d \times d$ discrete Fourier transform matrix
  - $B \in \{-1,1\}^{d \times d}$ is a random diagonal matrix with i.i.d. symmetric Bernoulli entries on its diagonal

Some Details

Definition

Let $m \in \{1, 2, \ldots, d\}$. A vector $\mathbf{x} \in \mathbb{C}^d$ will be called m-flat if its entries can be partitioned into at least $\left\lfloor \frac{d}{m} \right\rfloor$ contiguous blocks such that:

1. *Every block contains either m or $m + 1$ entries,*
2. *Every block contains at least one entry whose magnitude is $\geq \frac{\|\mathbf{x}\|_2}{2\sqrt{d}}$, and*

– The smaller m is, the flatter (and less sparse) $\mathbf{x}$ must be.

Some Details

Definition

Let $\epsilon \in (0,1)$, *and* $\mathcal{S} \subset \mathbb{C}^d$ *be finite. An* $m \times d$ *matrix* A *is a linear Johnson-Lindenstrauss embedding of* $\mathcal{S}$ *into* $\mathbb{C}^m$ *if*

$$(1-\epsilon)\| \mathbf{u} - \mathbf{v} \|_2^2 \leq \| A\mathbf{u} - A\mathbf{v} \|_2^2 \leq (1+\epsilon)\| \mathbf{u} - \mathbf{v} \|_2^2$$

holds $\forall \mathbf{u}, \mathbf{v} \in \mathcal{S} \cup \{\mathbf{0}\}$. *In this case we will say that* A *is a JL(*m*,*d*,*ϵ*)-embedding of* $\mathcal{S}$ *into* $\mathbb{C}^m$.

JL embeddings are closely related to the *Restricted Isometry Property (RIP)*. A matrix with the restricted isometry property can be used to construct a Johnson-Lindenstrauss embedding matrix.

Some Details

- $W = PFB$
- For any given $m \in \{1, 2, \ldots, d\}$, one can partition W into $\left\lfloor \frac{d}{m} \right\rfloor$ blocks of contiguous rows,

$$W = \left(\; W_1 \, W_2 \, \ldots \, W_{\lfloor \frac{d}{m} \rfloor} \; \right)^T.$$

- Each renormalized sub-matrix of W, $\sqrt{\frac{d}{m}} \cdot W_j$ is "almost" a random sampling matrix times a random diagonal Bernoulli matrix and behaves like a JL(m,d,ϵ)-embedding of our signal $\mathbf{x}$ into $\mathbb{C}^m$ (or $\mathbb{C}^{m+1}$).
- Each block of m consecutive entries of $W\mathbf{x}$ should have roughly the same ℓ_2-norm as one another.