Incorporating Edge Information in Fourier Reconstruction

with Applications to Magnetic Resonance Imaging

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An Illustration

- Let f be a compactly supported piecewise-smooth function on the real line. For example,

$$f(x) = \begin{cases} \frac{3}{2} & -\frac{3\pi}{4} \leq x < -\frac{\pi}{2} \\ \frac{7}{4} - \frac{x}{2} + \sin(x - \frac{1}{4}) & -\frac{\pi}{4} \leq x < \frac{\pi}{8} \\ \frac{11}{4}x - 5 & \frac{3\pi}{8} \leq x < -\frac{3\pi}{4} \\ 0 & \text{else} \end{cases}$$

- Given the first few Fourier coefficients,

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx, \quad |k| \leq N,$$

let us compute a partial sum Fourier reconstruction ...

An Illustration

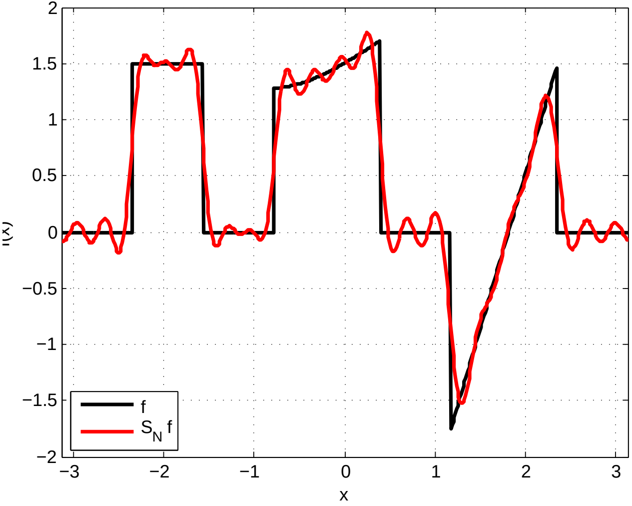


Figure: Reconstruction using 20 Fourier Coefficients

An Illustration

- Given $\hat{f}(k)$, how do we compute an approximation of the singular support of f?

$$[f](x) = \begin{cases} \frac{3}{2} & x = -\frac{3\pi}{4} \\ -\frac{3}{2} & x = -\frac{\pi}{2} \\ \frac{14+\pi}{8} - \sin\left(\frac{\pi+1}{4}\right) \approx 1.28 & x = -\frac{\pi}{4} \\ \sin\left(\frac{2-\pi}{8}\right) - \frac{28-\pi}{16} \approx -1.70 & x = \frac{\pi}{8} \\ \frac{33\pi}{32} - 5 \approx -1.76 & x = \frac{3\pi}{8} \\ 5 - \frac{33\pi}{16} \approx -1.48 & x = \frac{3\pi}{4} \\ 0 & \text{elsewhere} \end{cases}$$

where

$$[f](x) := f(x^+) - f(x^-).$$

An Illustration

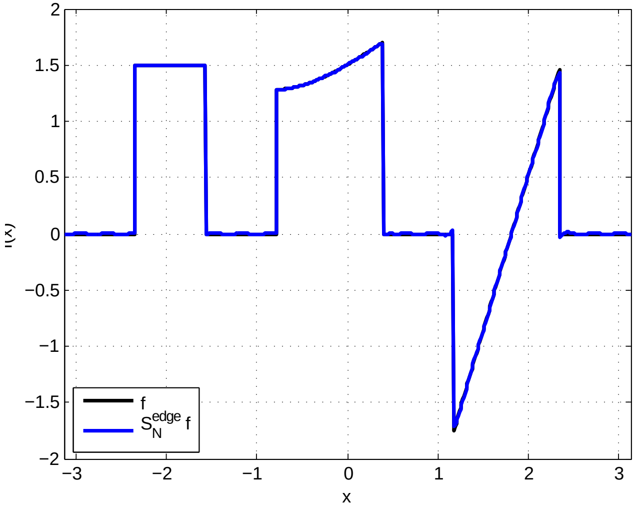
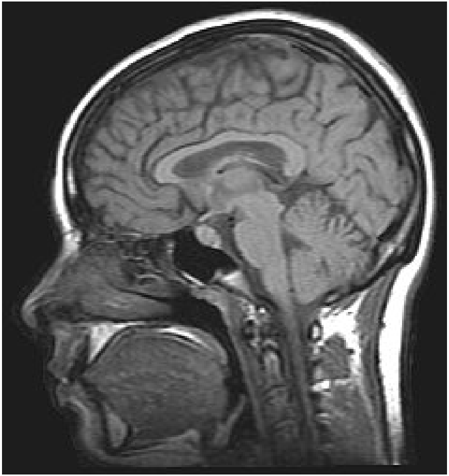


Figure: Reconstruction using 20 Fourier Coefficients and Edge Information

Motivating Application – Magnetic Resonance Imaging

- Physics of MRI dictates that the MR scanner collect samples of the Fourier transform of the scan image.
- In order to minimize scan time and improve patient comfort, we want to collect as few samples as possible.

Motivating Application – Magnetic Resonance Imaging



- Physics of MRI dictates that the MR scanner collect samples of the Fourier transform of the scan image.
- In order to minimize scan time and improve patient comfort, we want to collect as few samples as possible.
- Piecewise smooth nature of scan images degrades accuracy of many reconstruction schemes.

Outline

Concentration Method (Gelb, Tadmor)

- Approximate the singular support of f using the *generalized conjugate partial Fourier sum*

$$S_N^\sigma[f](x) = i \sum_{k=-N}^{N} \hat{f}(k)\,\mathsf{sgn}(k)\,\sigma\left(\frac{|k|}{N}\right) e^{ikx}$$

- $\sigma_{k,N}(\eta) = \sigma(\frac{|k|}{N})$ are known as *concentration factors* which are required to satisfy certain admissibility conditions.
- Under these conditions,

$$S_N^\sigma[f](x) = [f](x) + \mathcal{O}(\epsilon), \quad \epsilon = \epsilon(N) > 0 \text{ being small}$$

i.e., $S_N^\sigma[f]$ concentrates at the singular support of f.

Concentration Factors

Factor	Expression
Trigonometric	$\sigma_T(\eta) = \dfrac{\pi \sin(\alpha\,\eta)}{Si(\alpha)}$ $Si(\alpha) = \displaystyle\int_0^\alpha \frac{\sin(x)}{x}\,dx$
Polynomial	$\sigma_P(\eta) = -p\,\pi\,\eta^p$ p is the order of the factor
Exponential	$\sigma_E(\eta) = C\eta \exp\left[\frac{1}{\alpha\,\eta\,(\eta-1)}\right]$ C - normalizing constant α - order $C = \dfrac{\pi}{\int_{\frac{1}{N}}^{1-\frac{1}{N}} \exp\left[\frac{1}{\alpha\,\tau\,(\tau-1)}\right] d\tau}$

Table: Examples of concentration factors

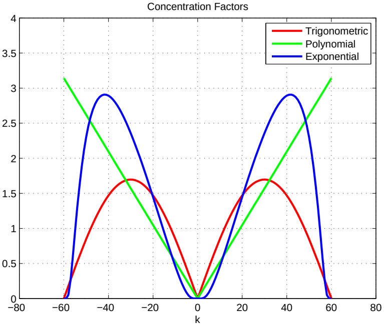


Figure:
Envelopes of Factors in k-space

Designing New Concentration Factors

$$S_N^\sigma[f](x) = ([f] * D_0^{\sigma,N})(x) + ([f'] * D_1^{\sigma,N})(x) + ([f''] * D_2^{\sigma,N})(x) + \dots$$

where

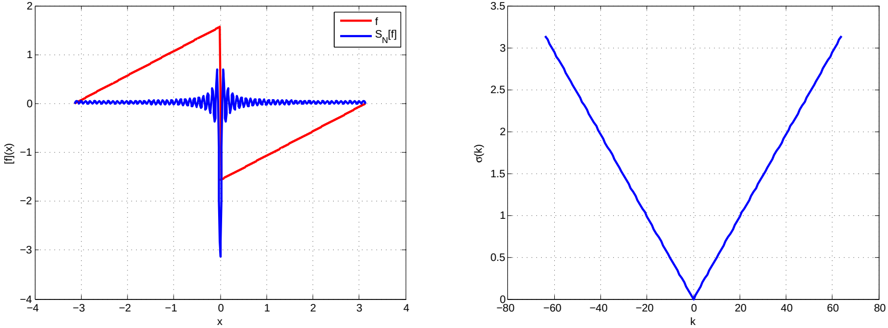
$$D_0^{\sigma,N}(x) := S_N^\sigma[r](x) = \frac{1}{2\pi} \sum_{0<|k|\leq N} \frac{\sigma\left(\frac{|k|}{N}\right)}{|k|} e^{ikx}.$$

- $D_0^{\sigma,N}$ is known as the characteristic response and is obtained by applying the concentration method to a ramp function.
- $D_0^{\sigma,N}$ (and consequently σ) completely defines the characteristics of the jump approximation.

Designing New Concentration Factors

$$\min_{\sigma} \quad \| D_0^{\sigma,N} \|_2$$

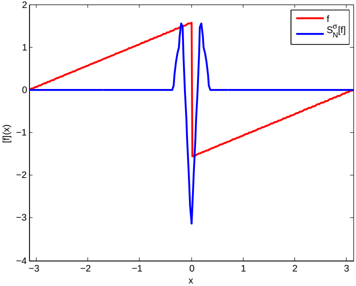
$$\text{subject to} \quad D_0^{\sigma,N}\Big|_{x=0} = 1$$



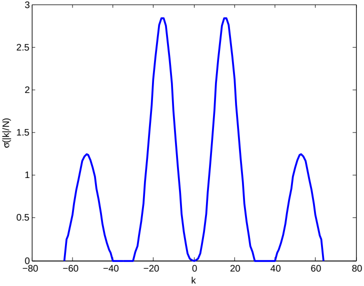
(a) Response (b) Concentration Factor

Designing New Concentration Factors

$$
\begin{aligned}
\min_{\sigma} \quad & \| D_0^{\sigma,N} \|_1 \\
\text{subject to} \quad & D_0^{\sigma,N}\Big|_{x=0} = 1 \\
& \Big| D_0^{\sigma,N}(x) \Big|_{|x| \geq .35} \leq 10^{-3} \\
& \sigma \geq 0,\ \sigma(1) = 0, \quad \sigma(\mathbb{K}) = 0
\end{aligned}
$$

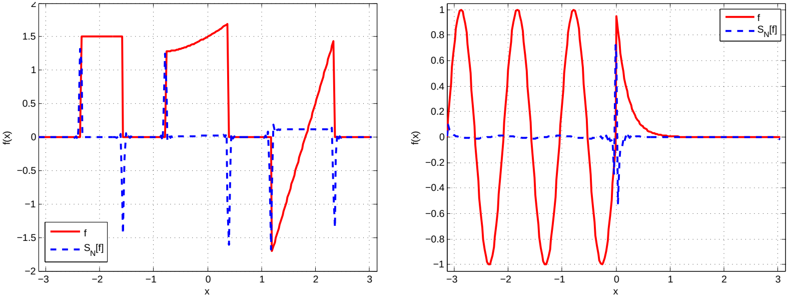


(c) Response



(d) Concentration Factor

Some Examples



(e) Trigonometric Factor (f) Exponential Factor

Figure: Jump Function Approximation, $N = 128$

Building an Edge Detector

- Exploit features of the characteristic response $D_0^{\sigma,N}$ such as mainlobe width and sidelobe structure.
- Formulate the problem in a statistical detection theoretic framework so that its performance can be quantified in the presence of noise.
- Resulting edge detector is a matched filter of the form

$$\to \mathcal{E}_{\text{true}} : M^T C_{\mathbf{V}}^{-1} \mathbf{Y} > \gamma$$

where Y is the vector of (possibly noisy) measurements from $S_N^\sigma[f]$, M is a template response, $C_{\mathbf{V}}$ is the covariance matrix and γ is a threshold.

Statistical Formulation of the Edge Detector

- Assume zero-mean, additive complex white Gaussian noise

$$\hat{\mathbf{g}}(k) = \hat{f}(k) + \hat{\mathbf{v}}(k) \quad \hat{\mathbf{v}}(k) \sim \mathcal{N}[0, \rho^2]$$

- Linearity of $S_N[f]$, i.e., $S_N^\sigma[g](x) = S_N^\sigma[f](x) + S_N^\sigma[\mathbf{v}](x)$
- Mean: $\mathsf{E}\left[\, S_N^\sigma[\mathbf{g}](x) \,\right] = S_N^\sigma[f](x)$
- Covariance: $(C_{\mathbf{v}})_{p,q}^{x_a,x_b} = \rho^2 \sum_l \sigma_p\left(|l|/N\right) \sigma_q\left(|l|/N\right) e^{il(x_a - x_b)}$
- The detection problem is

$$\begin{array}{lcll} \mathcal{H}_0: & \mathbf{Y} & = & \mathbf{V} & \sim \mathcal{N}[0, C_{\mathbf{V}}] \\ \mathcal{H}_1: & \mathbf{Y} & = & \alpha^1 \cdot M + \mathbf{V} & \sim \mathcal{N}[\alpha M, C_{\mathbf{V}}] \end{array}$$

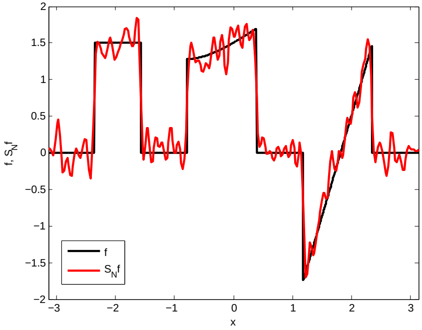
- Solve using Neyman-Pearson Lemma

$$\rightarrow \mathcal{H}_1: \quad \frac{Pr(\mathbf{Y}|\mathcal{H}_1)}{Pr(\mathbf{Y}|\mathcal{H}_0)} > \gamma$$

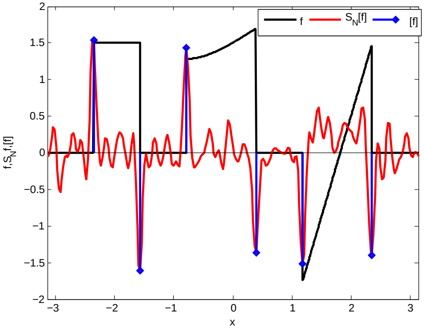
- Detector is a generalized matched filter
  $\rightarrow \mathcal{H}_1: \quad M^T C_{\mathbf{V}}^{-1} \mathbf{Y} > \gamma$
- False alarm mitigation using local maximum of statistic.

[1] α is the jump value.

Edge Detector Examples



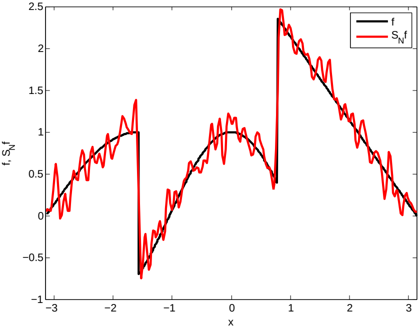
(a) Noisy Fourier Reconstruction



(b) Jump Detection

Figure: Edge Detection with Noisy Data, $N = 50, \rho = 0.02, 5-$point Trigonometric detector

Edge Detector Examples



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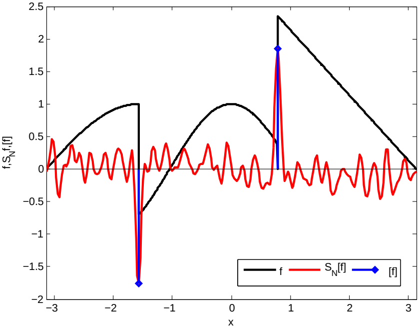


(b) Jump Detection

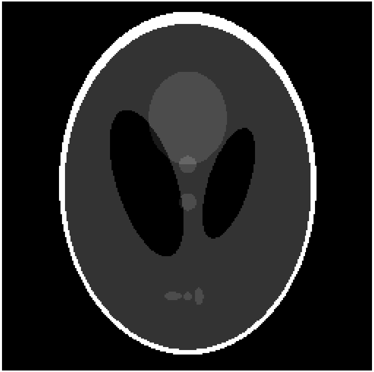
Figure: Edge Detection with Noisy Data, $N = 50, \rho = 0.02, 5-$point Trigonometric detector

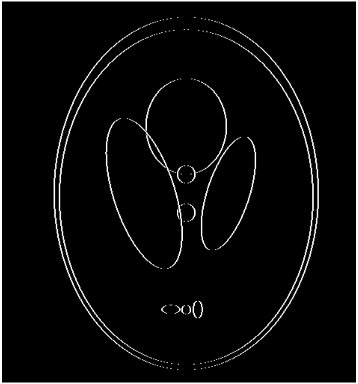
Two Dimensional Extensions

For images, apply the method to each dimension separately

$$S_N^\sigma[f](x(\bar{y})) = i \sum_{l=-N}^{N} \mathsf{sgn}(l)\, \sigma\left(\frac{|l|}{N}\right) \sum_{k=-N}^{N} \hat{f}_{k,l}\, e^{i(kx+l\bar{y})}$$

(overbar represents the dimension held constant.)





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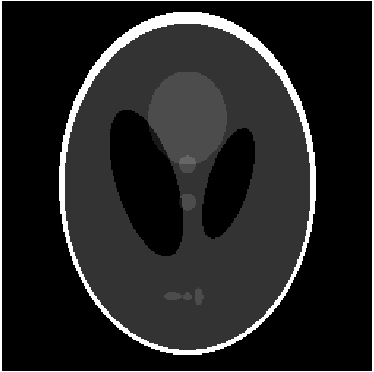
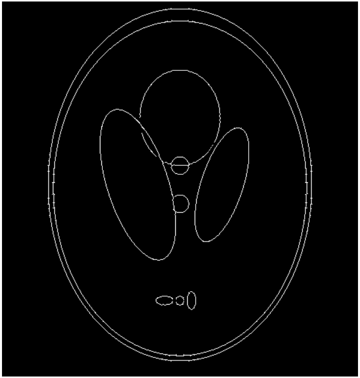




Iterative Formulation (with Wolfgang Stefan)

Recall,

$$S_N^\sigma[f](x) \approx ([f] * D_0^{\sigma,N})(x)$$

where

- $S_N^\sigma[f]$ is the jump approximation computed using the concentration method and concentration factor $\sigma(\eta)$.
- $D_0^{\sigma,N}$ is the template waveform generated by concentration factor $\sigma(\eta)$.

Problem Formulation

$$\min_p \quad \| \mathcal{F}\{Wp\}_{\omega_k} - \tilde{f}_{\omega_k} \|_2^2 + \lambda \| p \|_1$$

where W is a (banded) Toeplitz matrix of samples of $D_0^{\sigma,N}$ and $\tilde{f}_{\omega_k} := \hat{f}(\omega_k) \cdot i \cdot \mathsf{sgn}(\omega_k) \cdot \sigma\left(\frac{|\omega_k|}{N}\right)$.

Representative Examples

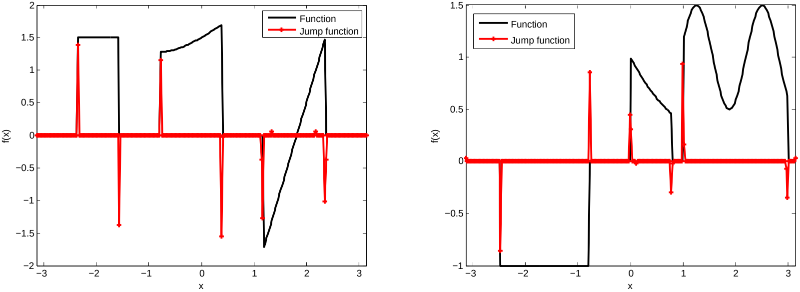


Figure: Jump detection – Iterative Formulation ($N = 40$, Exponential factor)

Representative Examples

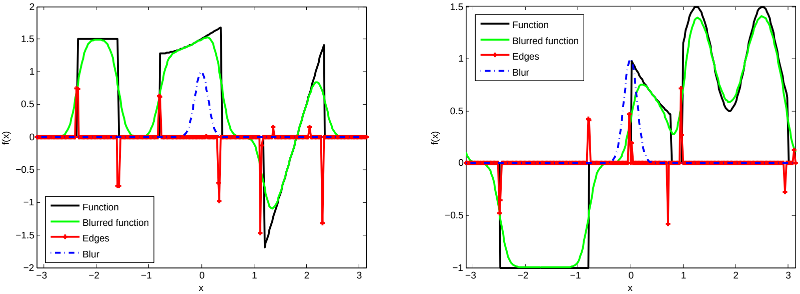


Figure: Jump detection – Iterative Formulation ($N = 128$, Exponential factor, Gaussian Blur)

Outline

Relating Fourier Data to Edge Information

Let f be periodic in $[-\pi, \pi)$ with a single jump at $x = \zeta$.

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\zeta^-} f(x)e^{-ikx}dx + \frac{1}{2\pi}\int_{\zeta^+}^{\pi} f(x)e^{-ikx}dx, \quad k \neq 0$$

Integrating by parts and using

$$[f](x) := f(x^+) - f(x^-),$$

we get

$$\hat{f}(k) = \frac{1}{2\pi}\left(\frac{[f](\zeta)}{ik} + \frac{[f'](\zeta)}{(ik)^2} + \frac{[f''](\zeta)}{(ik)^3} + ...\right) e^{-ik\zeta}, \quad k \neq 0$$

This is a relation between global Fourier measurements and local features (edges).

Fourier Coefficient Estimates

- Synthesize additional Fourier coefficients using

$$\hat{f}^{\text{est}}(k) = \sum_{p=1}^{n_p} \frac{[f](\zeta_p)}{2\pi i k}\, e^{-ik\zeta_p} + \mathcal{O}\left(\frac{1}{k^2}\right), \quad k \neq 0,$$

  (n_p is the number of jumps in the function)
  i.e., improve effective resolution of a reconstruction without collecting *any* additional data.
- High frequency coefficients may be synthesized at high SNRs.
- Applications in non-harmonic Fourier reconstruction, where k-space region of low sampling density can contain large re-gridding errors.

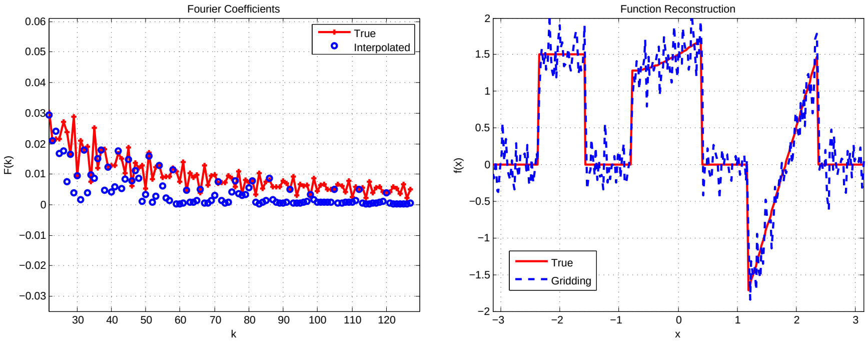
Improving the Effective Resolution of Fourier Reconstructions

Figure: Fourier Reconstruction of the Shepp-Logan Phantom using $N = 100$ Fourier Modes

Improving the Effective Resolution of Fourier Reconstructions

Figure: Same Reconstruction with a further $N = 100$ Modes estimated using edge information

Applications to Non-harmonic Fourier Reconstruction

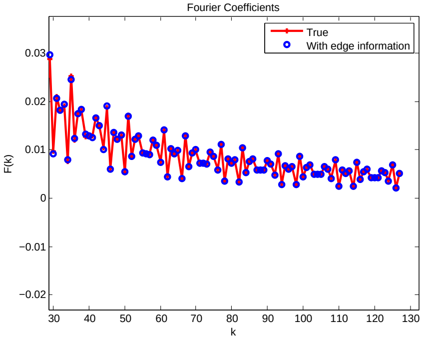


(a) Fourier coefficients – High modes

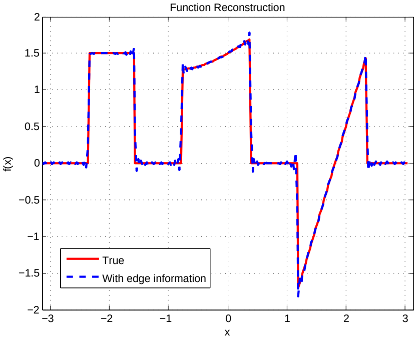
(b) Resulting reconstruction artifacts

Figure: Non-harmonic Fourier Reconstruction (Convolution Gridding, $N = 128$)

Applications to Non-harmonic Fourier Reconstruction



(a) Fourier coefficients – High modes



(b) Reconstruction - Using edge information

Figure: Reconstruction of the same test function using edge estimates

Edge Augmented Partial Fourier Sums

$$S_N^{\text{edge}} f(x) = \sum_{|k|\le N} \hat{f}(k) e^{ikx} + \sum_{|k|>N} \hat{f}^{\text{est}}(k) e^{ikx}$$

$$= \sum_{|k|\le N} \left[\hat{f}(k) - \hat{f}^{\text{est}}(k)\right] e^{ikx} + \sum_{k=-\infty}^{\infty} \hat{f}^{\text{est}}(k) e^{ikx}$$

$$= \sum_{|k|\le N} \left[\hat{f}(k) - \hat{f}^{\text{est}}(k)\right] e^{ikx} + \sum_{p=0}^{n_p} \frac{[f](\zeta_p)}{\pi} \sum_{k=1}^{\infty} \frac{\sin[k(x-\zeta_p)]}{k}$$

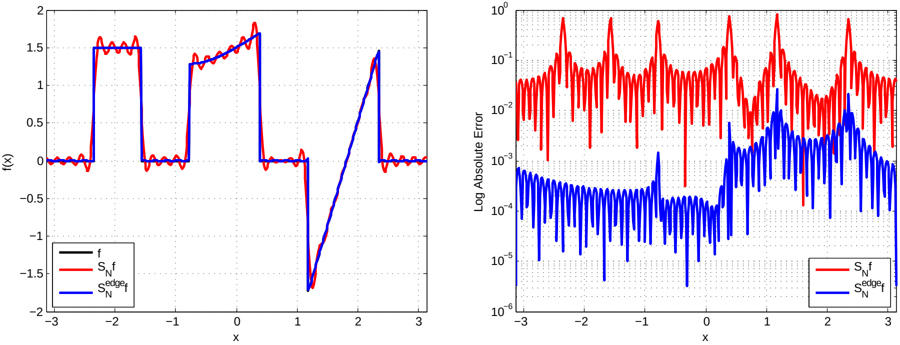
$$= \sum_{|k|\le N} \left[\hat{f}(k) - \hat{f}^{\text{est}}(k)\right] e^{ikx} + \sum_{p=0}^{n_p} [f](\zeta_p)\, r(x-\zeta_p)$$

where

$$r(x) = \frac{\pi - x}{2\pi} + \frac{1}{2} \textsc{Floor}\left[\frac{x}{2\pi}\right]$$

is the unit ramp (sawtooth) function.

Some Numerical Results

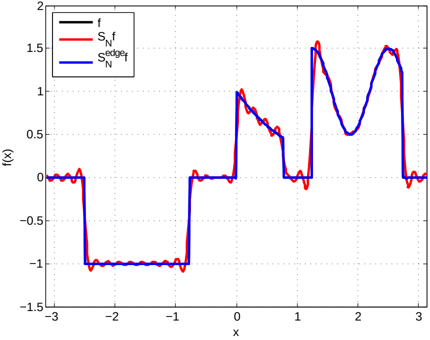


(a) Physical Reconstruction

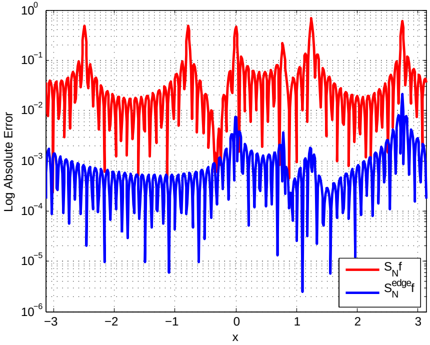
(b) Log Reconstruction Error

Figure: Function Reconstruction, $N = 32$

Some Numerical Results



(a) Physical Reconstruction



(b) Log Reconstruction Error

Figure: Function Reconstruction, $N = 32$

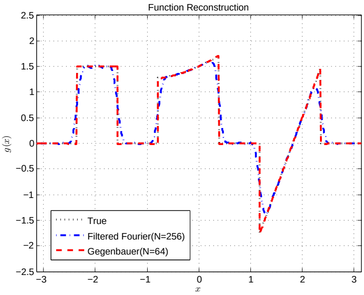
Spectral Re-projection

- Spectral reprojection schemes were formulated to resolve the Gibbs phenomenon. They involve reconstructing the function using an alternate basis, Ψ (known as a Gibbs complementary basis).
- Reconstruction is performed using the rapidly converging series

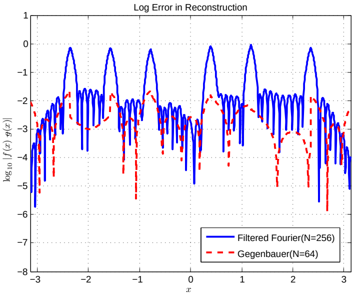
$$f(x) \approx \sum_{l=0}^{m} c_l \psi_l(x), \quad \text{where} \quad c_l = \frac{\langle S_N f, \psi_l \rangle_w}{\|\psi_l\|_w^2}$$

- Reconstruction is performed in each smooth interval. Hence, we require jump discontinuity locations
- High frequency modes of f have exponentially small contributions on the low modes in the new basis

Gegenbauer Reconstruction – Results



(a) Reconstruction



(b) Reconstruction error

Figure: Gegenbauer reconstruction

- Filtered Fourier reconstruction uses 256 coefficients
- Gegenbauer reconstruction uses 64 coefficients
- Parameters in Gegenbauer Reconstruction - $m = 2, \lambda = 2$

Summary and Future Directions

- The concentration method provides a flexible and robust framework for extracting edge information from Fourier data.
- Ongoing research is directed at developing truly two-dimensional formulations of the concentration method.
- Edge information extracted by this method can be used to improve reconstruction quality
  - Can be used to improve re-gridding performance in non-harmonic reconstructions
  - Can be used to synthesize high-mode Fourier coefficients to improve the effective resolution of scans
  - Can be used in spectral re-projection schemes to eliminate the Gibbs phenomenon.
- Applications in other fields for edge augmented Fourier approximations include study of wave propagation in layered or discontinuous media.

References

Edge Detection

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Fourier Reconstruction

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