1. Consider the following binary operations on \( \mathbb{N} = \{0, 1, 2, \ldots \} \)
\[
    a \oplus b = \max(a, b)
\]
\[
    a \otimes b = a + b.
\]

a. Does there exist \( e \in \mathbb{N} \) such that \( a \oplus e = a \) for every \( a \in \mathbb{N} \)? If yes, say what this element is (no need to explain). If not, explain why.
Yes: \( e = 0 \).

b. Does the equality \( a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c) \) hold for any \( a, b, c \in \mathbb{N} \)? If yes, show that it holds assuming \( b > c \). If not, give a counterexample.
It holds. Assuming \( b > c \) we have \( a \otimes (b \oplus c) = a \otimes b = a + b \) and \( (a \otimes b) \oplus (a \oplus c) = \max(a + b, a + c) = a + b \).

c. Which ring axiom does \((\mathbb{N}, \oplus, \otimes)\) fail to satisfy?
The axiom on the existence of the negatives does not hold: We have a “zero” element \( e = 0 \) (see 1.a. above), but for \( a = 2 \) there exists no \( a' \) such that \( a \oplus a' = e \).

2. For the following \( R \) and \( S \) say if \( S \) is a subring of \( R \) (without explanation):

a. \( R = \mathbb{Z} \) and \( S = 3\mathbb{Z} \).
Yes.

b. \( R = \mathbb{Z} \) and \( S = \mathbb{Z}_3 \).
No. \( \mathbb{Z}_3 \) is not a subset of \( \mathbb{Z} \).

c. \( R = \mathbb{Z} \) and \( S \) is the set of odd integers.
No. The set of odd numbers is not closed under addition.

d. \( R = \mathbb{Z} \) and \( S = \mathbb{Q} \).
No. \( \mathbb{Q} \) is not a subset of \( \mathbb{Z} \).