Exercises.

Say which are the units and which are the zero divisors in the following rings:

- $\mathbb{Z}_5$.
  
  Solution: Every element in $\mathbb{Z}_5 \setminus \{0\}$ is a unit. There are no zero divisors.

- $\mathbb{Z} \times \mathbb{Z}$.
  
  Solution: The units must be $(r, s)$ where both $r$ and $s$ are units in $\mathbb{Z}$. Therefore, the units are $(1, 1), (1, -1), (-1, 1)$ and $(-1, -1)$. The zero divisors are all the elements of the form $(r, 0)$ or $(0, r)$ where $r$ is a nonzero integer.

- $\mathbb{R} \times \mathbb{R}$.
  
  Solution: The units are all the elements of the form $(r, s)$ where $r$ and $s$ are nonzero. The elements of the form $(r, 0)$ or $(0, r)$ with nonzero $r$ are the zero divisors.

- $\mathbb{Z} \times \mathbb{Z}_6$.
  
  Solution: The units are $(1, [1]), (1, [5]), (-1, [1])$ and $(-1, [5])$. The zero divisors are all the elements of the form $(0, r)$ with $r \in \mathbb{Z}_6 \setminus \{0\}$, or $(r, 0)$ with $r$ any nonzero integer, or $(r, s)$ with $r \in \mathbb{Z}$ and $s \in \{[2], [3], [4]\}$.

- $M_2(\mathbb{R})$.
  
  Solution: The units are the matrices of nonzero determinant. The determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$. If $ad - bc \neq 0$ then the inverse matrix is $\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. If $ad - bc = 0$ then the matrix is a zero divisor (and in particular not invertible): $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
• $M_2(\mathbb{Z})$.
  Solution: The units are the matrices of determinant 1 or $-1$. The matrices of determinant 0 are the zero divisors.