Exercises.
Consider the ring \( \mathbb{Z} \times \mathbb{Z}_6 \). Say which of the following are subrings of this ring:

- \( \mathbb{Z} \times \mathbb{Z}_3 \).
- \( \mathbb{Z} \times [2] \cdot \mathbb{Z}_6 \).
- \( \mathbb{Z}_6 \times [2] \cdot \mathbb{Z}_6 \).
- \( 6\mathbb{Z} \times [2] \cdot \mathbb{Z}_6 \).
- \( 6\mathbb{Z} \times \{[0],[1],[2]\} \).
- \( 6\mathbb{Z} \times \{[0],[3]\} \).

Before solving the exercise, we shall start with the following statement:

**Theorem.**
Given rings \( A_1 \) and \( A_2 \) and subsets \( B_1 \subseteq A_1 \) and \( B_2 \subseteq A_2 \), \( B_1 \times B_2 \) is a subring of \( A_1 \times A_2 \) if and only if both \( B_1 \) is a subring of \( A_1 \) and \( B_2 \) is a subring of \( A_2 \).

**Proof.**
Recall that a subset \( S \subseteq R \) is a subring of \( R \) if the following holds

- \( S \) contains the zero.
- \( S \) is closed under addition.
- \( S \) is closed under multiplication.
• $S$ contains the negatives of its elements.

The zero element in $A_1 \times A_2$ is $(0,0)$. By the definition of Cartesian product, $(a, b)$ is in $B_1 \times B_2$ if and only if $a \in B_1$ and $b \in B_2$. By taking $a = 0$ and $b = 0$ one immediately obtains that $(0,0)$ is in $B_1 \times B_2$ if and only if $0 \in B_1$ and $0 \in B_2$.

Let $(a, b) \in B_1 \times B_2$. (By what we have just written, this is equivalent to saying $a \in B_1$ and $b \in B_2$.) $-(a, b) = (-a, -b)$, so $-(a, b) \in B_1 \times B_2$ if and only if $-(a, -b) \in B_1 \times B_2$ which holds if and only if $-a \in B_1$ and $-b \in B_2$.

Let $(a_1, b_1), (a_2, b_2) \in B_1 \times B_2$. $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$. Therefore $(a_1, b_2) + (a_2, b_2) \in B_1 \times B_2$ if and only if $(a_1 + a_2, b_1 + b_2) \in B_1 \times B_2$ which holds if and only if $a_1 + a_2 \in B_1$ and $b_1 + b_2 \in B_2$. Similarly, $(a_1, b_1) \cdot (a_2, b_2) \in B_1 \times B_2$ if and only if $(a_1 a_2, b_1 b_2) \in B_1 \times B_2$ which holds if and only if $a_1 a_2 \in B_1$ and $b_1 b_2 \in B_2$.

So $B_1 \otimes B_2$ satisfies the required axioms of a subring if and only if $B_1$ and $B_2$ satisfy those axioms.

**Solution to the exercise.**

$\mathbb{Z} \times \mathbb{Z}_3$ is not a subring of $\mathbb{Z} \times \mathbb{Z}_6$, because $\mathbb{Z}_3$ is not a subring of $\mathbb{Z}_6$. Remember!!!

We write $\mathbb{Z}_3 = \{[0], [1], [2]\}$ and $\mathbb{Z}_6 = \{[0], [1], [2], [3], [4], [5]\}$ for convenience, but this notation is misleading. The element $[0]$ in $\mathbb{Z}_3$ is NOT the same element as $[0]$ in $\mathbb{Z}_6$! The element $[0]$ in $\mathbb{Z}_3$ is the SET $3\mathbb{Z} = \{3n : n \in \mathbb{Z}\}$ whereas the element $[0]$ in $\mathbb{Z}_6$ is the SET $6\mathbb{Z} = \{6n : n \in \mathbb{Z}\}$.

$\mathbb{Z} \otimes 2\mathbb{Z}_6$ is by this logic a subring of $\mathbb{Z} \times \mathbb{Z}_6$. ($2\mathbb{Z}_6$ is a subring of $\mathbb{Z}_6$.) $\mathbb{Z}_6 \times 2\mathbb{Z}_6$ is not a subring, because $\mathbb{Z}_6$ is not a subring of $\mathbb{Z}$.

$6\mathbb{Z} \otimes 2\mathbb{Z}_6$ is a subring. ($6\mathbb{Z}$ is a subring of $\mathbb{Z}$.) $6\mathbb{Z} \times 2\mathbb{Z}_6$ is not a subring, because $\mathbb{Z}_6$ is not closed under addition in $\mathbb{Z}_6$ (and so not a subring of $\mathbb{Z}_6$).

$6\mathbb{Z} \times \{[0], [3]\}$ is a subring. The set $\{[0], [3]\}$ is a subring of $\mathbb{Z}_6$. It can also be written as $[3] \cdot \mathbb{Z}_6$. 

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