Exercises.

• Is it necessarily true that $a^b \equiv a^c \pmod{n}$ when $b \equiv c \pmod{n}$? [Check the case of $n = 3, a = 2, b = 1, c = 4$.]
  Solution: No. For example, $1 \equiv 4 \pmod{3}$, but $2^1 \neq 2^4 = 16 \equiv 1 \pmod{3}$.

• Let $R$ be a ring with identity. Prove that if for some $a, b, c \in R$, $ab = ba = 1$ and $ac = ca = 1$ then $b = c$.
  Solution: Look at $bac$. On the one hand, $bac = b(ac) = b \cdot 1 = b$. On the other hand $bac = (ba)c = 1 \cdot c = c$. Hence $b = c$.

• Prove that if $R$ is a ring with identity containing more than one element then $1 \neq 0$.
  Solution: Assume $1 = 0$. Let $a$ be a nonzero element in $R$. Since $1 = 0$, we have $a \cdot 1 = a \cdot 0$. However, $a \cdot 0 = 0$ whereas $a \cdot 1 = a$, which means that $a = 0$, contradictory to the choice of $a$.

• Explain why the following is not a ring: $(\mathbb{R}, \oplus, \otimes)$ where $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$. Say which axioms are satisfied and which are not.
  Solution: We have the axioms
  
  $a \oplus b = b \oplus a$
  
  $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
  
  $(a \otimes b) \otimes c = a \otimes (b \otimes c)$
\[(a \oplus b) \otimes c = a \otimes c \oplus b \otimes c\]

\[a \otimes (b \oplus c) = a \otimes b \oplus a \otimes c\]

However, there is no “zero” (i.e. there is no elements which is neutral to \(\oplus\)) and therefore there is no point in looking for “negatives”.

- Explain why the following is not a ring: \((\mathbb{R}_0^+, +, \cdot)\) where \(\mathbb{R}_0^+\) is the set of nonnegative real numbers. Say which axioms are satisfied and which are not.
  All the axioms are satisfied except the one on the existence of negatives. A nonzero element in this set has no negative in this set.

- Explain why the following is not a ring: \((\mathbb{R}, +, \otimes)\) where \(a \otimes b = \text{max}(a, b)\). Say which axioms are satisfied and which are not.
  Solution: All the axioms are satisfied except the “distributivity”, i.e. we do not have \(a \otimes (b+c) = a \otimes b + a \otimes c\). Take for example \(a = 2\) and \(b = c = 1\). Then \(2 \otimes (1+1) = \text{max}(2, 2) = 2\) whereas \(2 \otimes 1 + 2 \otimes 1 = \text{max}(2, 1) + \text{max}(2, 1) = 4\).