Home exercise.
Say in each of the following cases if the two-sided ideal $I < M_2(\mathbb{Z})$ can be proper. If so, give an example, and if not, explain why:

- $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \in I$.
- $\begin{pmatrix} 3 & 6 \\ 0 & 3 \end{pmatrix} \in I$.
- $\begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix} \in I$.

Solution.

- In this case $I$ is not proper, because by $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in I$, and then by column and row operations one can show that the identity matrix is in $I$, and so $I = M_2(\mathbb{Z})$.
- $I$ can be proper. For example, $I = M_2(3\mathbb{Z})$.
- This matrix is invertible, so $I$ is not proper. One can see that this matrix is invertible by the determinant which is 1.