Exercise. Say in the following cases whether \( h(x) \) is invertible in \( \mathbb{R}[x]/f(x) \) and if so then find the inverse:

- \( h(x) = x^3 - 6x^2 + 11x - 6 \) and \( f(x) = x^3 - x \).
- \( h(x) = x^3 - 3x^2 - 4x + 12 \) and \( f(x) = x^3 - x \).

Say whether the following polynomials are irreducible:

- \( x^2 + x + 1 \) in \( \mathbb{Z}_2[x] \).
- \( x^2 + x + 1 \) in \( \mathbb{Z}_3[x] \).
- \( x^2 + x + 1 \) in \( \mathbb{Q}[x] \).
- \( x^2 + x + 1 \) in \( \mathbb{R}[x] \).
- \( x^2 + x + 1 \) in \( \mathbb{C}[x] \)

First case:

\[
h(x) = f(x) - 6x^2 + 12x - 6
\]

We can proceed with dividing \( h(x) \) by \( x^2 - 2x + 1 \) instead of \(-6x^2 + 12x - 6\), because \(-6x^2 + 12x - 6 = -6(x^2 - 2x + 1)\).

\[
f(x) = (x + 2)(x^2 - 2x + 1) + 2x - 2
\]

For the same reason we shall divide \( x^2 - 2x + 1 \) by \( x - 1 \).

\[
x^2 - 2x + 1 = (x - 1)(x - 1)
\]
So \( \gcd(f(x), h(x)) = x - 1 \). Therefore \( h(x) \) is not invertible in \( F[x]/f(x) \).

There is also an alternative way of solving it:

\[
f(x) = x(x + 1)(x - 1)
\]

So \( \gcd(f(x), h(x)) \) is the product of all the polynomials among the three \( x \), \( x + 1 \) and \( x - 1 \) that divide \( f(x) \). The polynomials \( x \) and \( x + 1 \) do not divide \( f(x) \) because \( f(0) \) and \( f(-1) \) are nonzero, but \( x - 1 \) does, and so \( \gcd(f(x), p(x)) = x - 1 \).

Second case:

\[
h(x) = f(x) - 3x^2 - 3x + 12
\]

We shall divide \( f(x) \) by \( x^2 + x - 4 \)

\[
f(x) = (x - 1)(x^2 + x - 4) + 4x - 4
\]

Now we shall divide \( x^2 + x - 4 \) by \( x - 1 \):

\[
x^2 + x - 4 = (x + 2)(x - 1) - 2
\]

\[-2|x - 1\]

So \( \gcd(f(x), h(x)) \) is 1. Therefore \( h(x) \) is invertible in \( F[x]/f(x) \). We shall find the inverse:

\[
1 = -\frac{1}{2} \cdot (-2) = -\frac{1}{2}(x^2 + x - 4 - (x + 2)(x - 1))
\]

\[
= -\frac{1}{2}(x^2 + x - 4 - \frac{1}{4}(x + 2)(p(x) - (x - 1)(x^2 + x - 4)))
\]

\[
= -\frac{1}{2}((1 + \frac{1}{4}(x + 2)(x - 1))(x^2 + x - 4) - \frac{1}{4}(x + 2)f(x))
\]

\[
= -\frac{1}{2}((1 + \frac{1}{4}(x + 2)(x - 1))(\frac{1}{3}(h(x) - f(x)) - \frac{1}{4}(x + 2)f(x))
\]

\[
= \frac{1}{6}(1 + \frac{1}{4}(x + 2)(x - 1))h(x) + (-\frac{1}{6}(1 + \frac{1}{4}(x + 2)(x - 1)) - \frac{1}{4}(x + 2))f(x)
\]

So the inverse of \( h(x) \) in \( F[x]/f(x) \) is \([\frac{1}{6}(1 + \frac{1}{4}(x + 2)(x - 1))]\).

The polynomial \( f(x) = x^2 + x + 1 \) is irreducible in \( \mathbb{Z}_2[x] \) because it is of degree 2 and has no roots: \( f(0) = f(1) = 1 \).

The polynomial \( f(x) = x^2 + x + 1 \) is reducible in \( \mathbb{Z}_3[x] \) because it has a root: \( f(1) = 0 \).

The polynomial \( f(x) = x^2 + x + 1 \) is irreducible in \( \mathbb{Q}[x] \) because it is of degree 2 and has no roots: the two complex roots are \( \frac{-1 \pm 3i}{2} \) which are not real, never mind complex. For this reason, \( f(x) \) is also irreducible in \( \mathbb{R}[x] \). In \( \mathbb{C}[x] \) it is reducible because it has roots.