Exercise.
Show that $f : \mathbb{Z}_{mk} \rightarrow \mathbb{Z}_k \times \mathbb{Z}_m$, $f([n]_{mk}) = ([n]_k, [n]_m)$ is a homomorphism.

Explain why $f$ in this case is an isomorphism if and only if $\gcd(m, k) = 1$.
[Hint: If $\gcd(m, k) = 1$ one can use the Chinese remainder theorem to get injectivity. Since the sets are finite and of the same size, every injection is a bijection. If $\gcd(m, k) \neq 1$ then $mk > \text{lcm}(m, k)$, so one can find a nonzero element which is mapped to 0, and therefore $f$ is not injective.]

Find a homomorphism from $\mathbb{Z}_3$ to $\mathbb{Z}_6$. [Hint: Find the image of $[1]_3$ in $\mathbb{Z}_6$ by solving the system

$x \equiv 0 \pmod{2}$
$x \equiv 1 \pmod{3}$]

Solution.
Since $[a + b] = [a] + [b]$, we have

$f([a]+[b]) = f([a+b]) = ([a+b], [a]+[b]) = ([a]+[b], [a]+[b]) = ([a], [a])+([b], [b])$.

The multiplication property is demonstrated in a similar way.

If $\gcd(m, k) = 1$ then for any system

$x \equiv a \pmod{m}$
$x \equiv b \pmod{k}$

there exists a unique class of solutions in $\mathbb{Z}_{mk}$. In other words, for any $([a], [b]) \in \mathbb{Z}_m \times \mathbb{Z}_k$ there is a unique $[x] \in \mathbb{Z}_{mk}$ such that $f([x]) = ([a], [b])$. So $f(x)$ is injective. Since both sets are of the same finite size $mk$, any injection is a bijection. Since $f(x)$ is a homomorphism, it is an isomorphism.
Assume \( \gcd(m, k) \neq 1 \). Then \( t = \text{lcm}(m, k) < mk \). Consider \( f([t]) \). Since \( m|t \) and \( k|t \), \( f([t]) = ([0], [0]) \). However, \( [t] \neq [0] \), so \( f(x) \) is not injective.

By solving the system in the hint, one find that \([4]\) is the class of solutions in \( \mathbb{Z}_6 \). Therefore, the homomorphism we want is \( f : \mathbb{Z}_3 \rightarrow \mathbb{Z}_6, f([n]) = [4n] \). One can verify that \( f([n] \cdot [m]) = [4n] \cdot [4m] = [4 \cdot 4] \cdot [nm] = [4] \cdot [nm] = [4nm] = f([nm]). \)