Exercise.
Say about the following if they are homomorphisms or not:

- $f : M_2(\mathbb{R}) \to \mathbb{R} \times \mathbb{R}$, $f \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = (a, b)$.
- $f : \mathbb{Z}_4 \to \mathbb{Z}_2$, $f([x]) = [x]$.
- $f : \mathbb{Z}_2 \to \mathbb{Z}_4$, $f([0]) = [0]$ and $f([1]) = [1]$.

Solution.

- This is not a homomorphism. One can notice for instance that if it were a homomorphism, it would be an epimorphism because the function is injective. Then it would send the identity to the identity, but the identity matrix is mapped to $(1, 0)$. A more direct argument is to consider the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We have $A \cdot A = \text{Id}_{2 \times 2}$, so $f(A \cdot A) = (1, 0)$ but $f(A) \cdot f(A) = (0, 1) \cdot (0, 1) = (0, 1)$.

- This map is a homomorphism. Let $[a]_4 + [b]_4 = [c]_4$. Then $c \equiv a + b \pmod{4}$. Therefore $c - (a + b) \in 4\mathbb{Z}$ which means that $c - (a + b) \in 2\mathbb{Z}$ (a multiple of 4 is always a multiple of 2) and so $c \equiv a + b \pmod{2}$. Hence $[a]_2 + [b]_2 = [c]_2$. Now, $f([a]_4) = [a]_2$ etc., so this way we get $f([a]_4) + f([b]_4) = f([c]_4)$. The multiplication property is demonstrated in the same manner.

- This is not a homomorphism. For example $f([1] + [1]) = f([0]) = [0]$ whereas $f([1]) + f([1]) = [1] + [1] = [2]_4 \neq [0]_4$.

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