Exercise.
In a commutative ring $R$ with identity, prove that if $a$ is a zero divisor and $b$ is a unit then $a \not| b$. Give an example where $b|a$.

Solution: Assume $a|b$. Then $b = a \cdot c$ for some $c$. Since $a$ is a zero divisor, there exists $a' \neq 0$ such that $a' \cdot a = 0$. So $a'b = a'(ac) = (a'c)a = 0 \cdot c = 0$, so $b$ is a zero divisor. However zero divisors cannot be units, contradiction.

Consider $R = \mathbb{Z}[x]$. Say whether the following congruences if they hold:

- $x^2 + 4x^5 \equiv 6 - x^2 + 2x^3 + 4x^5 \pmod{2}$.
- $x^2 + 4x^5 \equiv 6 - x^2 + 2x^3 + 4x^5 \pmod{x}$.
- $x^2 + 4x^5 \equiv 6 - x^2 + 2x^3 + 4x^5 \pmod{3}$.
- $x^2 + 4x^5 \equiv 6 - x^2 + 2x^3 + 4x^5 \pmod{4}$.

Solution: The difference between the polynomials is $6 - 2x^2 + 2x^3$, so the answers are

- Yes, because every coefficient is a multiple of 2.
- No, because the free coefficient is nonzero.
- No, because there coefficients which are not multiples of 3.
- No, because there are coefficients which are not multiples of 4.

Let $m, n \in \mathbb{Z}$. What is $m\mathbb{Z} \cap n\mathbb{Z}$? Say whether the following statements hold:

- $12 \in 4\mathbb{Z} \cap 6\mathbb{Z}$.
• $6 \in 4\mathbb{Z} \cap 6\mathbb{Z}$.

• $24 \in 4\mathbb{Z} \cap 6\mathbb{Z}$.

Solution: An element is in $m\mathbb{Z} \cap n\mathbb{Z}$ if and only if it is both a multiple of $m$ and of $n$. Such an element is a multiple of the least common multiple of the two, i.e. $m\mathbb{Z} \cap n\mathbb{Z} = \text{lcm}(m, n)\mathbb{Z}$. Now, lcm$(4, 6) = 12$, so

• Yes.

• No.

• Yes.