Exercises.
Say which of the following is a field, an integral domain which is not a field, or a ring with identity which is not an integral domain:

- \( \mathbb{Z} \).
- \( \mathbb{Z}_6 \).
- \( \mathbb{Z}_5 \).
- \( \mathbb{R} \).
- \( \mathbb{Z} \otimes \mathbb{Z} \).
- \( \mathbb{R} \otimes \mathbb{R} \).
- \( \mathbb{R}[x] \).
- \( \mathbb{Z}_6[x] \).

Solution.

- \( \mathbb{Z} \) is an integral domain.
- \( \mathbb{Z}_6 \) is not an integral domain (it contains zero divisors).
- \( \mathbb{Z}_5 \) is a field.
- \( \mathbb{R} \) is a field.
• \( \mathbb{Z} \otimes \mathbb{Z} \) is not an integral domain.
• \( \mathbb{R} \otimes \mathbb{R} \) is not an integral domain.
• \( \mathbb{R}[x] \) is an integral domain (but not a field).
• \( \mathbb{Z}_6[x] \) is not an integral domain.

**Polynomial rings.**
We introduced the ring of polynomials \( R[x] \) in one variable over a ring \( R \). We shall see some of its properties. Recall that a polynomial is an expression of the form

\[
f(x) = c_0 + c_1 x + \cdots + c_n x^n
\]

where \( n \in \mathbb{N} \) and \( c_0, \ldots, c_n \in R \). Multiplication and addition follow from \( R \). Equality between polynomials means coefficient-wise equality. The degree of a nonzero polynomial \( f(x) \), denoted by \( \deg(f(x)) \) is the maximal power of \( x \) with nonzero coefficient in \( f(x) \). Note that the minimal possible degree is 0.

**Examples.**
The degree of \([1] + [4]x + [3]x^2 \) in \( \mathbb{Z}_6[x] \) is 2. The degree of \( \frac{1}{3} + x^6 - 3x^2 \) is 6 in \( \mathbb{Q}[x] \).

**Question.**
Given a ring (which is not an integral domain) \( R \), is it true that the degree of \( \deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x)) \)?

**Answer.**
No. Take for example \( R = \mathbb{Z}_6 \), \( f(x) = [3]x \) and \( g(x) = 1 + [2]x \). Then \( f(x)g(x) = [3]x + [0]x^2 = [3]x \). \( \deg(f(x)g(x)) = 1 \neq 2 = \deg(f(x)) + \deg(g(x)) \).

**Exercise.**
Show that if \( R \) is an integral domain and \( f(x), g(x) \in R[x]\setminus\{0\} \) then \( \deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x)) \).

**Solution.**
Let \( n = \deg(f(x)) \) and \( m = \deg(g(x)) \). Then \( f(x) = a_0 + \cdots + a_n x^n \) and \( g(x) = b_0 + \cdots + b_m x^m \) where \( a_n, b_m \neq 0 \). Now, \( h(x) = f(x)g(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + \cdots + (a_{n-1}b_m + a_nb_{m-1})x^{n+m-1} + a_nb_{m}x^{m+n} \). In particular, the coefficient of \( x^{m+n} \) in \( h(x) \) is \( a_nb_m \). Since \( a_n, b_m \neq 0 \) and \( R \) is an integral domain, \( a_nb_m \) is nonzero. This is the term of highest degree in \( h(x) \), so \( m + n \) is the degree of \( h(x) \).

**Remark.**
Note that if \( R \) is not an integral domain, we still have the inequality

\[
\deg(fg) \leq \deg(f) + \deg(g).
\]
Exercise.
Prove that if $R$ is an integral domain then $R[x]$ is an integral domain.

Proof.
Let $f(x), g(x) \in R[x] \setminus \{0\}$. Write $n = \deg(f(x))$ and $m = \deg(g(x))$, $f(x) = a_0 + \cdots + a_n x^n$ and $g(x) = b_0 + \cdots + b_m$ as in the previous exercise and $h(x) = f(x)g(x)$. Then the coefficient of $x^{n+m}$ in $h(x)$ is nonzero, which means that $h(x)$ is not zero. Consequently there are no zero divisors in $R[x]$.

Question.
Given an integral domain (or even a field) $R$, can $R[x]$ be a field?

Answer.
No. The degree of 1 is 0. The degree of $f(x) = x$ is 1. For any $g(x) \in R[x]$, $\deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x)) \geq \deg(f(x)) = 1 > 0$, which means that $f(x)g(x)$ is different from 1 for any $g(x)$. This means that $f(x)$ is not invertible, and as a result $R[x]$ is not a field.

Question.
Given an integral domain $R$, what are the units of $R[x]$?

Answer.
As we saw before, if the degree of $f(x)$ is nonzero then $f(x)$ is not a unit. Therefore all the units in $R[x]$ are degree 0 elements, which are elements in $R$. So the units in $R[x]$ are exactly $R^\times$ (i.e. the units in $R$).

Question.
Given a ring $R$ which is not an integral domain $R$, are the units of $R[x]$ necessarily $R^\times$?

Answer.
No. Take $R = \mathbb{Z}_4$. Then $([1] + [2])x \cdot ([1] + [2])x = [1] + [4]x + [4]x^2 = [1]$. So $[1] + [2]x$ is a unit of degree 1, and not in $R^\times$.

Exercise.

• Prove that in $\mathbb{Z}_n[x]$ we have $(x + [1])^n = x^n + [1]$ when $n$ is prime.

• Take $S = x \cdot \mathbb{R}[x] = \{x \cdot f(x) : f(x) \in \mathbb{R}[x]\}$. Is that a subring of $\mathbb{R}[x]$? Describe the polynomials in $S$ by their behavior when substituting $x = 0$.

• Let $S$ be the set of polynomials in $\mathbb{Z}[x]$ where the free coefficient (the coefficient of $x^0 = 1$) is even. Is $S$ a subring of $\mathbb{Z}[x]$? What about the set of polynomials with odd free coefficient?

• Take $S$ to be the set of polynomials $f(x)$ in $\mathbb{R}[x]$ such that $f(1) = 0$. Is $S$ a subring of $\mathbb{R}[x]$? What about the set of polynomials with $f(1) = 1$?