Relations.
A relation from $A$ to $B$ (or between $A$ and $B$) is a subset $R$ of $A \times B$. For convenience we often write $aRb$ instead of $(a, b) \in R$. For a given relation $R$ between $A$ and $B$ we define its domain to be $A$, and denote it by $\text{Dom}(R)$ and its image to be

$$\text{Im}(R) = \{ b \in B : \exists_{a \in A} aRb \}. $$

There is also the co-domain or “range” $\text{CoDom}(R) = B$ and the co-image

$$\text{CoIm}(R) = \{ a \in A : \exists_{b \in B} aRb \}. $$

Examples.
Take $A = B = \mathbb{R}$ and $R$ to be the relation $\leq$, i.e. $R = \{(a, b) \in \mathbb{R}^2 : a \leq b \}$. This is why we write $aRb$, which in this case turns into $a \leq b$.

Take $A = \{1, 2, 3\}$, $B = \{a, b, c\}$ and $R = \{(1, a), (1, b), (2, b), (3, b)\}$. Since the number of elements involved is finite, we can describe the relation using the following diagram

```
1  →  a
   ↘
2  →  b
   ↗
3  ↘
c.
```

The image in this case is $\{a, b\}$ and the co-image is $A$.

Functions.
We call a relation \( R \) between \( A \) and \( B \) a “function” if for any \( a \in A \) there exists exactly one \( b \in B \) such that \( aRb \). In this case, we write \( b = R(a) \), and \( b \) is called the image of \( a \) under \( R \), \( a \) is a co-image of \( b \) and \( R \) maps \( a \) to \( b \). The expression \( R : A \to B \) means that \( R \) is a function from \( A \) to \( B \).

**Examples.** Take \( A = \{1, 2, 3\} \) and \( B = \{a, b, c\} \). Then

1  →  a
2  →  b
3  ↘  c

is not a function.

1  →  a
2  →  b
3  ↗  c

is a function.

1  →  a
2  →  b
3  ↘  c

is not a function.

If \( A = B = \mathbb{R} \), then \( f : \mathbb{R} \to \mathbb{R} \) which is defined by \( f(x) = x^2 \) is a function. Note that \( f(x) = \frac{1}{x} \) is not a function from \( \mathbb{R} \) to \( \mathbb{R} \) because it is not defined at 0. However, if one takes \( A \) to be \( \mathbb{R} \setminus \{0\} \) then \( f : A \to \mathbb{R} \) defined by \( f(x) = \frac{1}{x} \) is a function.

**Types of functions.**

Let \( f : A \to B \).

- We say that \( f \) is 1 − 1 (one to one) or “injective” (or an injection) if for any \( b \in B \) there exists at most one \( a \in A \) (a “unique” \( a \in A \)) for which \( f(a) = b \).
• We say that $f$ is “onto” or “surjective” (or a surjection) if for any $b \in B$ there exists $a \in A$ such that $f(a) = b$.

• We say that $f$ is “bijective” (or a bijection) if it is $1-1$ and onto.

**Examples.**

1 $\rightarrow a$

2 $\rightarrow b$

3 $\rightarrow c$

is neither $1-1$ nor onto.

1 $\rightarrow a$

2 $\rightarrow b$

3 $\rightarrow c$

is onto but not $1-1$.

1 $\rightarrow a$

2 $\rightarrow b$

is a bijection.

1 $\rightarrow a$

2 $\rightarrow b$

$c$

is $1-1$ but not onto.

**Fact.**

If $A$ and $B$ are finite sets and there exists a bijection $f : A \rightarrow B$ then $A$ and $B$ are
of the same size. If there exists an injection \( f : A \to B \) then \(|A| \leq |B|\) and if there exists a surjection \( f : A \to B \) then \(|A| \geq |B|\).

**Theorem.**

If \( A \) and \( B \) are finite sets of the same size then a function \( f : A \to B \) then the following are equivalent:

- \( f \) is 1-1.
- \( f \) is onto.
- \( f \) is bijective.

The theorem does not hold if \( A \) and \( B \) are infinite (in which case the word “size” requires clarification). For example, take \( A = B = \mathbb{N} \) and \( f : A \to B \) defined by \( f(x) = x + 1 \). It is injective but not surjective. If one defines \( f(0) = 0 \) and for every \( n > 0, f(n) = n - 1 \), then \( f \) is surjective but not injective.

**Exercises.**

- Say about the following functions \( f : \mathbb{R} \to \mathbb{R} \) if they are injective, surjective, bijective or nothing of that sort: \( f(x) = x^2, f(x) = x^3, f(x) = 4x, f(x) = 1, f(x) = x + 1 \).

- Now consider the functions \( f : \mathbb{N} \to \mathbb{N} \) with the same formulas as above and answer the same question.

- In the second question, say in all the cases the function is not surjective, what the image is.