Abstract Algebra I - Lecture 29

Adam Chapman

Department of Mathematics, Michigan State University, East Lansing, MI 48824

Definition.
A left ideal that is also a right ideal is called a two-sided ideal. For a two-sided ideal $I$ in $R$ we write $I \triangleleft R$.

Remark.
In a commutative ring every left ideal is a two-sided ideal. In that case we simply say ideal.

Exercise.
Show that in $M_2(\mathbb{R})$ there is no nonzero proper left ideal that is also a right ideal.

Solution.
Let $I$ be a nonzero two-sided ideal in $M_2(\mathbb{R})$. Take a nonzero matrix $M$ in $I$. This means there is a nonzero entry $m_{i,j}$ in row $i$ and column $j$ for some $i$ and $j$. Multiply $M$ from the left by a matrix with 1 in row $i$ and column $i$ and zeros elsewhere. Then multiply from the right by a matrix with 1 in row $j$ and column $j$ and zeros elsewhere. Multiply by $\frac{1}{m_{i,j}} \text{Id}$. What we get is a matrix with 1 in line $i$ and column $j$ and zeros elsewhere. This matrix is in $I$.

Now, by multiplying from the left and from the right by matrices inducing elementary row and column operations, we can move 1 anywhere we want. In particular, we can place it on the diagonal. We can also add up such matrices, and we remain in $I$, which means that we obtain that the identity matrix is in $I$, and so $I$ is not proper.

Example.
If $I \triangleleft R$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I$ with $b \neq 0$ then $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in I$.

Then $\begin{pmatrix} \frac{1}{b} & 0 \\ 0 & \frac{1}{b} \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in I$.

Email address: adam1chapman@yahoo.com (Adam Chapman)
Then \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in I \) and \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in I \).

So \( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in I \), which means \( I = R \).

**Fact.**
For any field \( F \) and any integer \( n \), \( M_n(F) \) contains no nonzero proper two-sided ideals.

**Exercise.**
Find a two-sided ideal in \( R = M_2(\mathbb{Z}) \).

**Solution.**
Take \( I = M_2(2\mathbb{Z}) \). This is the set of all the matrices with even entries. If \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is in \( R \) and \( B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \) is in \( I \) then \( \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} \) is a matrix with even entries and hence in \( I \). Similarly \( BA \) is a matrix with even entries and hence in \( I \).

**Home exercise.**
Say in each of the following cases if the two-sided ideal \( I \triangleleft M_2(\mathbb{Z}) \) can be proper. If so, give an example, and if not, explain why:

- \( \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \in I. \)
- \( \begin{pmatrix} 3 & 6 \\ 0 & 3 \end{pmatrix} \in I. \)
- \( \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix} \in I. \)