Exercise.
Show that if $f : R \to S$ and $g : S \to T$ are homomorphisms then $g \circ f : R \to T$ is a homomorphism.

Proof.
Let $a, b \in R$. Then $f(a + b) = f(a) + f(b)$. Now $g \circ f(a + b) = g(f(a) + f(b)) = g(f(a)) + g(f(b)) = g \circ f(a) + g \circ f(b)$. In a similar manner, one can show that $g \circ f(ab) = (g \circ f(a))(g \circ f(b))$.

Remark.
Since the composition of bijections is a bijections, from the last exercise we conclude that the composition of isomorphisms is an isomorphism.

Proposition.
If $f : R \to S$ is an isomorphism then also the inverse function $f^{-1} : S \to R$ is an isomorphism.

Proof.
The inverse function is obviously also bijective. What is left to show is that it satisfies $f^{-1}(a + b) = f^{-1}(a) + f^{-1}(b)$ and $f^{-1}(ab) = f^{-1}(a)f^{-1}(b)$. Write $f^{-1}(a) = a'$ and $f^{-1}(b) = b'$. Since $f$ is a homomorphism, $f(a' + b') = f(a') + f(b') = a + b$. Take $f^{-1}(\ldots)$ of both sides: $a' + b' = f^{-1}(a + b)$, and $a' = f^{-1}(a)$ and $b' = f^{-1}(b)$, so $f^{-1}(a + b) = f^{-1}(a) + f^{-1}(b)$. In a similar way one can show that $f^{-1}(ab) = f^{-1}(a)f^{-1}(b)$.

Definition.
We say that $R$ is isomorphic to $S$ and write $R \cong S$ if there exists an isomorphism $f : R \to S$. By the previous proposition, this relation is symmetric. The identity map on $R$ is an isomorphism, so this relation is reflexive. By the remark above, the relation is also transitive. Therefore $\cong$ is an equivalence relation for rings.

Exercise.
Prove that if \( f : R \to S \) is an epimorphism and \( R \) is commutative then \( S \) is commutative.

**Proof.**
Let \( a, b \in S \). Then there exist \( c, d \in R \) such that \( f(c) = a \) and \( f(d) = b \). Then
\[
ab = f(c)f(d) = f(cd) = f(dc) = f(d)f(c) = ba.
\]

**Question.**
Let \( R \) and \( S \) be two rings with identity. Let \( f : R \to S \) be an epimorphism. Is it true that if \( S \) contains zero divisors then \( R \) contains zero divisors as well? Is it true that if \( R \) contains zero divisors then \( S \) contains zero divisors?

**Answer.**
No and no. Take \( f : \mathbb{Z} \to \mathbb{Z}_6, f(n) = [n]_6, \) and \( g : \mathbb{Z}_6 \to \mathbb{Z}_3, g([n]_6) = [n]_3. \) The rings \( \mathbb{Z} \) and \( \mathbb{Z}_3 \) do not contain zero divisors, but \( \mathbb{Z}_6 \) does, even though both \( f \) and \( g \) are epimorphisms.

**Exercise.**
Let \( f : R \to S \) be an isomorphism. Prove that \( R \) contains zero divisors if and only if \( S \) contains zero divisors.

**Proof.**
Assume \( R \) contains zero divisors \( ab = 0, a \neq 0 \) and \( b \neq 0. \) Since \( a \neq 0 \) and \( f \)
is injective, \( f(a) \neq 0. \) Similarly, \( f(b) \neq 0. \) Now \( f(a)f(b) = f(ab) = f(0) = 0, \) so \( f(a) \) and \( f(b) \) are zero divisors. Since \( f \) is an isomorphism, \( f^{-1} \) is also an isomorphism, and we can repeat the argument for \( f^{-1} \) to show that if \( S \) contains zero divisors then \( R \) contains zero divisors.

**Remark.**
In a similar way one can show that if \( f : R \to S \) is a monomorphism and \( R \)
contains zero divisors then \( S \) contains zero divisors.

**Exercise.**
Let \( f : R \to S \) be an epimorphism of rings with identity and \( r \in R^\times. \) Show that \( f(r) \in S^\times. \)

**Proof.**
There is an inverse \( r^{-1} \) of \( r \) in \( R. \) Since \( f \) is an epimorphism of rings with identity,
\( f(1) = 1. \) Now \( 1 = f(1) = f(r \cdot r^{-1}) = f(r) \cdot f(r^{-1}), \) and similarly \( 1 = f(r^{-1}) \cdot f(r), \)
so \( f(r)^{-1} = f(r^{-1}). \)

**Exercise.**
Let \( f : R \to S \) be a homomorphism. Show that \( \text{Im}(f) \) is a subring of \( S. \)