Sets.

A set is a well defined collection of elements. Those elements are called “members of the set” or “elements of the set”. Given a set $S$ and an element $x$ we write $x \in S$ if $x$ is a member of the set $S$. One can also say that $x$ belongs to $S$ or contained in $S$, and that $S$ contains $x$, or even $x$ is in the set $S$. By a well defined set we mean that every element $x$ is either in $S$ or not. If it is not then we write $x \notin S$.

Members of the set.

We use the curly brackets to indicate the elements of a given set. For example

$$S = \{1, 2, 3, 4\}$$

is the set consisting of the four integers, 1, 2, 3 and 4. One can therefore write $1 \in S$, and $5 \notin S$.

A set can contain a finite number of elements or an infinite number of elements. For example, the set above contains four elements. The set of integers, denoted by $\mathbb{Z}$, contains infinitely many elements. Similarly, we use $\mathbb{R}$ for the set of real numbers (e.g. $\pi, \sqrt{2}$), $\mathbb{Q}$ for the set of rational numbers (e.g. $\frac{1}{2}, 0.3$) and $\mathbb{N}$ for the set of natural numbers - i.e. all the nonnegative integers.

The members of a set can be numbers, and in most examples in this course we use sets of numbers, but the elements of a set can be objects of any kind, including sets. For example, the set

$$S = \{\{1, 2\}, \{3, 4, 5\}\}$$

is a set consisting of two elements, both of them are sets:

$$\{1, 2\}, \{3, 4, 5\}.$$
Another example is the set
\[ \mathbb{Z}_2 = \{2\mathbb{Z}, 2\mathbb{Z} + 1\} \]
which contains two elements, the set of even integers \(2\mathbb{Z}\) and the set of odd integers \(2\mathbb{Z} + 1\). Later on in this course we shall give this set, and the more general sets \(\mathbb{Z}_n\), a ring structure.

**Subsets.**

Given two sets, \(A\) and \(B\), we say that \(A\) is a subset of \(B\) if every element of \(A\) is also an element of \(B\), and we write \(A \subseteq B\). For example, \(\{1, 2, 3\} \subseteq \mathbb{Z}\).

There is an equality \(A = B\) if \(A \subseteq B\) and \(A \supseteq B\). For example, if \(A = \{1, 2, 4\}\) and \(B\) is the set of all divisors of 4 then \(A = B\).

Given a set \(B\), one can define a subset by adding a condition on the element. We write \(\{x \in B : \ldots\}\) and the condition appears on the right-hand side of the colon. For example, \(A = \{x \in \mathbb{Z} : x^2 - 3x + 2 = 0\}\) is the set \(A = \{1, 2\}\).

We can also define a set by a formula in the same manner. For example,
\[ \{2x : x \in \mathbb{Z}\} \]
is the set of all even integers, often denoted by \(2\mathbb{Z}\).

**Operators.**

The sign \(\cup\) stands for the union of two sets, i.e. the set whose elements are the elements of those two sets put together. For example, \(\{1, 2, 3\} \cup \{2, 3, 4\} = \{1, 2, 3, 4\}\). Note that we do not write the same element twice (in this case we did not write 2 and 3 twice).

The sign \(\cap\) stands for the intersection of two sets, the set containing only the elements which are contained in both sets. For example, \(\{1, 2, 3\} \cap \{2, 3, 4\} = \{2, 3\}\). Note that for any two sets, \(A \cap B \subseteq A, B\) and \(A, B \subseteq A \cup B\).

The sign \(\setminus\) stands for the removal of those elements in the first set that also appear in the second set. For example, \(\{1, 2, 3\} \setminus \{2, 3, 4\} = \{1\}\).

**The empty set.**

The empty set is the set that does not contain any element. It can be written as \(\{\}\) but in the literature it is more commonly denoted by \(\emptyset\). This set is a subset of any other set, i.e. \(\emptyset \subseteq A\) for any set \(A\).

**Connections to Logic.**


The union \( A \cup B \) can be described as \( \{x : x \in A \lor x \in B\} \).

The intersection \( A \cap B \) can be described as \( \{x : x \in A \land x \in B\} \).

The removal \( A \setminus B \) can be described as \( \{x : x \in A \land x \notin B\} \).

Inclusion of sets \( A \subseteq B \) can be understood by

\[
\forall x \in A \ x \in B
\]

or

\[
x \in A \implies x \in B.
\]

Equality of sets \( A \) and \( B \) can be understood by

\[
x \in A \iff x \in B.
\]

**Cartesian product of sets.**

Given two sets, \( A \) and \( B \), the Cartesian product \( A \times B \) is defined to be the set of ordered pairs

\[
A \times B = \{(a, b) : a \in A, b \in B\}.
\]

Note that in ordered pairs, unlike in sets, the order of the elements is important, and we allow multiplicity. For example, if \( A = \{1, 2\} \) and \( B = \{2, 3, 4\} \) then

\[
A \times B = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}.
\]

We use the word “product” because when taking about finite sets, the size of \( A \times B \) is exactly the product of the sizes of \( A \) and \( B \).

Another example is the Cartesian product of vector spaces. If \( A = \mathbb{R}^m \) and \( B = \mathbb{R}^n \) then \( A \times B = \mathbb{R}^{m+n} \).

The notion of ordered pairs can be extended to ordered \( n \)-tuples for any positive integer \( n \). For example,

\[
\{0, 1\} \times \{2\} \times \{3, 4, 5\} = \{(0, 2, 3), (0, 2, 4), (0, 2, 5), (1, 2, 3), (1, 2, 4), (1, 2, 5)\}.
\]

**Exercises.**
• Explain by logical terms why \( \emptyset \) is a subset of any other set.

• List all the elements of the set \( \{ x \in \mathbb{Z} : x^2 < 5 \} \).

• How many elements does \( \{ \emptyset \} \) contain?

• Describe the intersection of the set of all multiples of 2 (which we denote by \( 2\mathbb{Z} \)) and the set of all multiples of 3 (which we denote by \( 6\mathbb{Z} \)).