Review for midterm exam 1.

Exercise.

• Find $\gcd(11, 30)$ using Euclid’s algorithm.
• Express $\gcd(11, 30)$ as $11m + 30n$ for some integers $m, n$.
• Find $[30]^{-1}$ in $\mathbb{Z}_{11}$.

Solution.

\[
\begin{align*}
30 &= 2 \cdot 11 + 8 \\
11 &= 8 + 3 \\
8 &= 2 \cdot 3 + 2 \\
3 &= 2 + 1 \\
1 &= 2 - 1
\end{align*}
\]

so $\gcd(11, 30) = 1$.

Now

\[
\begin{align*}
3 &= 11 - 8 = 11 - (30 - 2 \cdot 11) = 3 \cdot 11 - 30 \\
2 &= 8 - 2 \cdot 3 = 30 - 2 \cdot 11 - 2 \cdot (3 \cdot 11 - 30) = 3 \cdot 30 - 8 \cdot 11
\end{align*}
\]
1 = 3 − 2 = 3 \cdot 11 − 30 − (3 \cdot 30 − 8 \cdot 11) = 11 \cdot 11 − 4 \cdot 30

so \( m = 11 \) and \( n = −4 \).

Therefore \([11]^{-1} = [11]\) in \( \mathbb{Z}_{30} \) and \([30]^{-1} = [−4]\) in \( \mathbb{Z}_{11} \).

**Exercise.**
Find \( \gcd(22, 33) \) and \( \text{lcm}(22, 33) \). Is \([22]\) invertible in \( \mathbb{Z}_{33} \)?

**Solution.**
The prime factorization of 22 is 22 = 11 \cdot 2 and of 33 is 33 = 11 \cdot 3. Therefore \( \gcd(22, 33) = 11 \) and \( \text{lcm}(22, 33) = 11 \cdot 2 \cdot 3 = 66 \).

[22] is not invertible in \( \mathbb{Z}_{33} \) because \( \gcd(22, 33) \neq 1 \).

**Exercise.**
Consider the set \( R = \{ x \in \mathbb{R} : x \geq 1 \} \) and the binary operators \( + \) and \( \otimes \), where \( + \) is the usual addition and \( \otimes \) is defined by \( a \otimes b = \max(a, b)^{\min(a, b)} \). For each of the following statements say if it holds:

- \( a \otimes b = b \otimes a \) for any \( a, b \in R \).
- \( (a \otimes b) \otimes c = a \otimes (b \otimes c) \) for any \( a, b, c \in R \).
- There exists \( e \in R \) such that \( a \otimes e = e \otimes a = a \).
- \( (a + b) \otimes c = (a \otimes c) + (b \otimes c) \)

**Solution.** Since \( \min(a, b) = \min(b, a) \) and \( \max(a, b) = \max(b, a) \), we have \( a \otimes b = b \otimes a \).

The second statement does not hold. Take \( a = b = 2 \) and \( c = 3 \). Then \( (a \otimes b) \otimes c = 2^2 \otimes 3 = 4 \otimes 3 = 4^3 = 64 \) whereas \( a \otimes (b \otimes c) = 2 \otimes 3^2 = 2 \otimes 9 = 9^2 = 81 \).

The third statement is true. Take \( e = 1 \). By the definition of \( R \), every \( a \in R \) satisfies \( a \geq 1 \), so \( \max(a, 1) = a \) and \( \min(a, 1) = 1 \), which means \( a \otimes 1 = a^1 = a \).

The fourth statement does not hold. Take \( a = b = 1 \) and \( c = 3 \). Then \( (a + b) \otimes c = 2 \otimes 3 = 3^2 = 9 \) whereas \( (a \otimes c) + (b \otimes c) = 3^1 + 3^1 = 6 \).