1. Solve the following differential equation

\[ \frac{dy}{dx} = \sqrt{y} \cos^2 \sqrt{y} \]

with the initial condition \( y(0) = 1 \).

2. State whether the following sequences of numbers \( \{a_n\}_{n \geq 1} \) converge or diverge. If they converge, find the limit.

   (a) \( a_n = \sqrt{2n + 1} \)

   (b) \( a_n = (-3)^n \)

3. Compute the integral and simplify as much as possible

\[ \int \frac{\sqrt{9 - w^2}}{w^2} \, dw \]

(Hint: Use the identity \( 1 + \cot^2 \theta = 1/\sin^2 \theta \) and compute also \( (\cot \theta)' \))

4. Determine whether the following improper integral converges. State the method you have used.

\[ \int_1^\infty \frac{1}{\sqrt{e^x - x}} \, dx \]

5. Compute the integral

\[ \int_0^1 \ln x \, dx \]
Solutions:

1. Separating the variables yields

\[
\int \frac{dy}{\sqrt{y} \cos^2 \sqrt{y}} = \int dx
\]

Substituting \( u = \sqrt{y} \), \( du = \frac{1}{2\sqrt{y}} dy \) we obtain

\[
\int \frac{dy}{\sqrt{y} \cos^2 \sqrt{y}} = 2 \int \sec^2 u \ du = 2 \tan u + C = 2 \tan \sqrt{y} + C
\]

and

\[2 \tan \sqrt{y} = x + C.\]

Solving for \( y \) we get

\[y = \left( \tan^{-1} \left( \frac{x}{2} + C \right) \right)^2\]

The initial condition \( y(0) = 1 \) leads to

\[\pm 1 = \tan^{-1}(C) \text{ and } C = \pm \tan(1)\]

so we get two solutions.

2. (a) We may consider the function \( f(x) = \sqrt{2x + 1} \) and compute its limit for \( x \to +\infty \). We first take the logarithm \( g(x) = \ln f(x) = \ln \frac{\ln (2x + 1)}{x} \) and use l'Hopital’s rule. We have

\[
\lim_{x \to +\infty} \frac{2}{2x + 1} = 0
\]

and therefore \( \lim_{x \to +\infty} g(x) = 0 \), i.e. \( \lim_{x \to +\infty} f(x) = 1 \). Hence

\[
\lim_{n \to +\infty} \sqrt{2n + 1} = 1.
\]

(b) This sequence has no limit since \( |a_n| \to +\infty \) and signs of the numbers \( a_n \) alternate.

3. Substitute \( w = 3 \sin \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \) so that \( dw = 3 \cos \theta \ d\theta \) and \( \sqrt{9 - w^2} = 3 \cos \theta \). We obtain

\[
\int \frac{\sqrt{9 - w^2}}{w^2} dw = \int \frac{3 \cos \theta}{9 \sin^2 \theta} \cdot 3 \cos \theta \ d\theta = \int \cot^2 \theta \ d\theta.
\]

Note that \( \sin^2 \theta + \cos^2 \theta = 1 \) implies

\[1 + \cot^2 \theta = \frac{1}{\sin^2 \theta}\]
and that

\[(\cot \theta)' = -\frac{1}{\sin^2 \theta}.
\]

Hence

\[
\int \cot^2 \theta \, d\theta = \int (-1 - (\cot \theta)') \, d\theta = -\theta - \cot \theta + C
\]

and

\[
\int \frac{\sqrt{9 - w^2}}{w^2} \, dw = -\sin^{-1}\left(\frac{w}{3}\right) - \cot\left(\sin^{-1}\left(\frac{w}{3}\right)\right) + C
\]

\[
= -\sin^{-1}\left(\frac{w}{3}\right) - \frac{\sqrt{9 - w^2}}{w} + C.
\]

(the simplification in the last step was done with the usual "triangle drawing trick")

4. We remark first that the integrand is a positive function. So we have to figure out whether the improper integral equals $+\infty$ or whether it remains finite.

Solution with the Limit Comparison Test: Because $e^x$ grows faster than $x$ we choose

\[g(x) = \frac{1}{\sqrt{e^x}} = e^{-x/2}, \quad f(x) = \frac{1}{\sqrt{e^x - x}}.
\]

We check whether $f(x)$ and $g(x)$ grow at the same rate:

\[
\lim_{x \to +\infty} \frac{g(x)}{f(x)} = \lim_{x \to +\infty} \frac{\sqrt{e^x - x}}{e^x} = \lim_{x \to +\infty} \frac{1 - x}{e^x} = 1
\]

Hence we can use the limit comparison test with the above choice of $g(x)$.

We compute

\[
\int_1^{\infty} g(x) \, dx = \int_1^{\infty} e^{-x/2} \, dx = \lim_{b \to +\infty} (-2e^{-b/2}) + 2e^{-1/2} = 0 + 2e^{-1/2} = \frac{2}{\sqrt{e}}
\]

Hence, by the limit comparison test, the improper integral $\int_1^{\infty} f(x) \, dx$ converges.

Solution with the Direct Comparison Test: We note that for $x \geq 1$

\[x \leq \frac{1}{2} e^x \Rightarrow -x \geq -\frac{1}{2} e^x
\]

This can be seen as follows: The left hand side $f(x) = x$ satisfies $f(1) = 1$ and it has slope 1. On the other hand, the function $g(x) = \frac{1}{2} e^x$ satisfies $g(1) = \frac{e}{2} > 1$ and also $g'(x) = \frac{e^x}{2} > 1$. So we must have $f(x) \leq g(x)$ for all $x \geq 1$. We estimate now for $x \geq 1$:

\[\sqrt{e^x - x} \geq \sqrt{e^x - \frac{1}{2} e^x} = \sqrt{\frac{1}{2} e^x} = \frac{1}{\sqrt{2}} e^{x/2}
\]
and therefore
\[ \frac{1}{\sqrt{e^x - x}} \leq \frac{\sqrt{2}}{e^{x/2}} = \sqrt{2}e^{-x/2}. \]

Hence
\[
\int_1^{\infty} \frac{1}{\sqrt{e^x - x}} \, dx \leq \int_1^{\infty} \sqrt{2}e^{-x/2} \, dx
\]
\[= \sqrt{2} \lim_{b \to +\infty} \int_1^{b} e^{-x/2} \, dx
\]
\[= 2\sqrt{2} \lim_{b \to +\infty} (e^{-1/2} - e^{-b/2})
\]
\[= 2\sqrt{2}e^{-1/2} < +\infty.\]

This means that the integral \( \int_1^{\infty} \frac{1}{\sqrt{e^x - x}} \, dx \) converges.

5. We first compute the integral of the logarithm function using integration by parts:
\[ \int \ln x \, dx = \int 1 \cdot \ln x \, dx = x \ln x - x + C. \]

Then
\[
\int_0^1 \ln x \, dx = \lim_{a \to 0} \int_a^1 \ln x \, dx
\]
\[= \lim_{a \to 0} \left[ x \ln x - x \right]^1_a
\]
\[= -1 - \lim_{a \to 0} (a \ln a)
\]
\[= -1 - \lim_{a \to 0} \frac{\ln a}{1/a}
\]
\[= -1 - \lim_{a \to 0} \frac{1}{1/a^2}
\]
\[= -1.
\]