Supplemental Material for Section 7.4:
Exponential Functions

In this section the general exponential function, $a^x$ is defined and some of its properties are presented. This document contains material that should have been included in the text. Recall that $a^x$ has already been defined for many values of $x$. For example $a^{\frac{1}{2}}$ denotes $\sqrt{a}$. In fact $a^r$ has been defined for any rational number $r = \frac{m}{n}$ by $a^r = (\sqrt[n]{a})^m$. Because only positive numbers have $n^{th}$ roots, it’s natural to assume that $a > 0$ which we do. To see how to use the function $e^x$ to define $a^x$, note that for $r$ a rational number, $a^r = e^{\ln a^r} = e^{r \ln a}$. The definition of $a^x$ is accomplished by simply insisting that the same formula holds for any real number $x$; not just rational numbers.

**Definition.** Let $a > 0$ and let $x$ be any real number. Then $a^x = e^{x \ln a}$.

The basic laws of exponents will now be established from this definition using the properties of the function $e^x$ and the laws of logarithms. We first note that for $a > 0$ and any real number $x$, $\ln a^x = \ln e^{x \ln a} = x \ln a$ by the definition of $a^x$ and that $\ln e^y = y$ for any number $y$.

**Theorem 1.** Let $a > 0$, $b > 0$ and let $x$ and $y$ be real numbers. Then:

1. $a^x a^y = a^{x+y}$
2. $(a^x)^y = a^{xy}$
3. $a^x b^y = (ab)^x$

**Proof.**

1. $a^x a^y = e^{x \ln a} e^{y \ln a} = e^{(x+ y) \ln a} = a^{x+ y}$. The last equality is the definition of $a^{x+ y}$.

2. $(a^x)^y = e^{y \ln a^x} = e^{y (x \ln a)} = e^{(xy) \ln a} = a^{xy}$. The last equality is the definition of $a^{xy}$.

3. $a^x b^y = (e^{x \ln a})(e^{y \ln b}) = e^{x (\ln a+ \ln b)} = e^{y (\ln a+ \ln b)} = e^{x \ln(ab)} = (ab)^x$. The last equality is the definition of $(ab)^x$.

Using $-x$ for $y$ in 1. we obtain that $a^x a^{-x} = a^0 = e^{0 \ln a} = e^0 = 1$. Consequently $e^{-x} = \frac{1}{e^x}$. From this fact and 1. we obtain the familiar formula $\frac{a^x}{a^y} = a^{x-y}$.