Supplemental Material for Section 11.2: Convergence of an Infinite Series

The notion of a convergent infinite series is central to the rest of the material in Chapter 8. The idea is given a sequence of numbers \( a_0, a_1, a_2, a_3 \ldots \) investigate the sum of these numbers; that is, \( a_0 + a_1 + a_2 + a_3 + \ldots \). What should be clear is that if such a sum exists, then it can be approximated by \( a_0 + a_1 + \cdots + a_k = \sum_{n=0}^{k} a_n \). For each positive integer \( k \). So as \( k \) increases, the approximation gets better and better. The formal definition makes all of this precise using the notion of limit of a sequence.

**Definition.** Let \( \{a_n\} \) be a sequence (of terms). Then \( \sum a_n \) converges means that the sequence \( \{\sum_{n=0}^{k} a_n\} \) has a limit that is a number. Otherwise we say that \( \sum a_n \) diverges.

The sequence \( \{s_k\} \) defined by \( s_k = \sum_{n=0}^{k} a_n \) is called the sequence of partial sums of the infinite series \( \sum a_n \). So said another way \( \sum a_n \) converges means the sequence \( \{s_k\} \) has a numerical limit. If \( \sum a_n \) converges, then we let \( \sum_{n=0}^{\infty} a_n = \lim_{k \to \infty} \sum_{n=0}^{k} a_n \).

To better understand the meaning of “\( \sum a_n \) converges”, let \( \{a_n\} \) be a sequence of terms. Define a function \( f \) on the infinite interval \([0, \infty)\) by \( f(x) = a_n \) for \( x \) in the interval \([n, n + 1)\) for each positive integer \( n \). The graph of \( f \) is given in Figure 1.

![Figure 1: Graph of \( y = f(x) \)](image-url)
For any positive integer $k$, $s_k = \sum_{n=0}^{k} a_n = \int_{0}^{k} f(x) \, dx$. Thus $\sum a_n$ converges; that is, $\lim_{k \to \infty} s_k$ exists and is a nimber is equivalent to $\int_{0}^{\infty} f(x) \, dx$ converges. When each $a_n \geq 0$, $\sum a_n$ converges means that the area under the graph of $y = f(x)$ is finite.